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Initial–boundary value problems for conservation laws with source terms and the Degasperis–Procesi equation ☆

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Abstract

We consider conservation laws with source terms in a bounded domain with Dirichlet boundary conditions. We first prove the existence of a strong trace at the boundary in order to provide a simple formulation of the entropy boundary condition. Equipped with this formulation, we go on to establish the well-posedness of entropy solutions to the initial–boundary value problem. The proof utilizes the kinetic formulation and the averaging lemma. Finally, we make use of these results to demonstrate the well-posedness in a class of discontinuous solutions to the initial–boundary value problem for the Degasperis–Procesi shallow water equation, which is a third order nonlinear dispersive equation that can be rewritten in the form of a nonlinear conservation law with a nonlocal source term.

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Contents

1. Introduction

In this article we consider scalar conservation laws with source terms in a bounded open subset $\Omega \subset \mathbb{R}^d$ with C^2 boundary:

$$\partial_t u + \operatorname{div}_x A(u) = S(t,x,u), \quad (t,x) \in Q := (0,T) \times \Omega, \tag{1}$$

where $T > 0$ is a fixed final time and the flux function $A \in C^2$ satisfies the genuine nonlinearity condition

$$\mathcal{L}\bigl(\bigl\{\xi \bigm| \tau + \zeta \cdot A'(\xi) = 0\bigr\}\bigr) = 0, \quad \text{for every } (\tau,\zeta) \neq (0,0), \tag{2}$$

where $\mathcal{L}$ is the Lebesgue measure.

The source term satisfies the following conditions:

$$S \in L^\infty(Q \times \mathbb{R}), \qquad S(t,x,\cdot) \in C^1(\mathbb{R}), \qquad \bigl|S(t,x,u) - S(t,x,v)\bigr| \leqslant C|u-v|, \tag{3}$$

where the last two conditions hold for a.e. $(t,x) \in Q$ and $C > 0$ is a constant.

As usual, we only deal with entropy solutions, namely those that fulfill in the sense of distributions on Q the inequality

$$\partial_t \eta(u) + \operatorname{div}_x q(u) - \eta'(u) S(t,x,u) \leqslant 0 \tag{4}$$

for every convex C^2 function η and related entropy flux defined by

$$q' = A'\eta'.$$

We are interested in the well-posedness in L^∞ of the initial–boundary value problem for (1), in which case we impose the initial data

$$u(0,\cdot)=u_0\in L^\infty(\Omega) \tag{5}$$

and the Dirichlet boundary data

$$u|_\Gamma=u_b\in L^\infty(\Gamma), \tag{6}$$

where $\Gamma:=(0,T)\times\partial\Omega$. Of course, this Dirichlet condition has to be interpreted in an appropriate sense (see below) and this in turn requires an entropy solution to possess boundary traces (which herein will be understood in a strong sense).

A BV well-posedness theory for conservation laws with Dirichlet boundary conditions was first established by Bardos, le Roux, and Nédélec [2], and later extended by Otto [24] to the L^∞ setting, for which boundary traces do not exist in general, a fact that complicates significantly the notion of solution and the proofs. For genuinely nonlinear fluxes and domains whose boundaries satisfy a mild regularity assumption, Vasseur [31] showed that L^∞ entropy solutions always have traces at the boundaries. Similar results hold without imposing a genuine nonlinearity condition, cf. Panov [25,26] and Kwon and Vasseur [17]. Consequently, for genuinely nonlinear fluxes, the L^∞ case can be treated as in [2], i.e., the more complicated notion of entropy solution used by Otto can be avoided, see Kwon [16].

To define traces on the boundary Γ we use the concept of a "regular deformable boundary" (see for instance Chen and Frid in [3]). For any domain Ω with $\mathcal{C}^2$ boundary, there exists at least one $\partial\Omega$-regular deformation. Given any open subset $\hat{K}$ of $\partial\Omega$, we refer to a mapping $\hat{\psi}:[0,1]\times\hat{K}\to\overline{\Omega}$ as a $\hat{K}$-regular deformation provided it is a $\mathcal{C}^1$ diffeomorphism and $\hat{\psi}(0,\cdot)\equiv I_{\hat{K}}$ with $I_{\hat{K}}$ denoting the identity map over $\hat{K}$. Let us now define the set $K:=(0,T)\times\hat{K}$ and the function $\psi(s,\hat{z}):=(\hat{t},\hat{\psi}(s,\hat{x}))$ where $\hat{z}:=(\hat{t},\hat{x})\in K$. Then, obviously, $\psi(s,\hat{z})$ is K-regular deformation with respect to Γ. Let us denote by $\hat{n}_s$ the unit outward normal field of the deformed boundary $\hat{\psi}(\{s\}\times\partial\Omega)$. We also write $n_s=(0,\hat{n}_s)$ and $n=(0,\hat{n})$. Notice that $\hat{n}_s$ converges strongly to $\hat{n}$ when s goes to 0.

Our first main result is the following theorem.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^d$ be a regular open set with $\mathcal{C}^2$ boundary. Assume that* (3) *holds and that the flux function $A\in C^2(\mathbb{R})$ verifies* (2)*. Consider any function $u\in L^\infty((0,T)\times\Omega)$ obeying* (1) *and* (4) *in $(0,T)\times\Omega$. Then*

- *there exists $u^\tau\in L^\infty((0,T)\times\partial\Omega)$ such that for every Γ-regular deformation ψ and every compact set $K\Subset\Gamma$ there holds*

$$\operatorname*{ess\,lim}_{s\to 0}\int_K\left|u\big(\psi(s,\hat{z})\big)-u^\tau(\hat{z})\right|d\sigma(\hat{z})=0,$$

where $d\sigma$ denotes the volume element of $(0,T)\times\partial\Omega$;
- *there exists $u^\tau\in L^\infty(\Omega)$ such that for every compact set $K\Subset\Omega$ there holds*

$$\operatorname*{ess\,lim}_{t\to 0}\int_K\left|u(t,x)-u^\tau(x)\right|dx=0.$$

In particular, the trace u^τ is unique and, for any continuous function F, $F(u)$ also possesses a trace and

$$\left[F(u)\right]^\tau = F\left(u^\tau\right).$$

The proof of this theorem is found in Section 2. More precisely, in this section we prove the first part of Theorem 1.1. The second part can be proved using the same method, so we omit the details. The method of proof follows along the lines of Vasseur [31], utilizing the blow-up method, the kinetic formulation developed by Lions, Pethame, and Tadmor [18], and a version of the averaging lemma (see Perthame and Souganidis [29]). An alternative proof can be given using Panov's H-measure approach, cf. [26], which moreover requires a genuine nonlinearity condition that is less restrictive than (2).

Having settled the existence of strong boundary traces, we can now turn to the choice of entropy boundary condition. Instead of working with the original condition due to Bardos, le Roux, and Nédélec [2], we shall instead employ the following equivalent boundary condition introduced by Dubois and LeFloch [11], which is well defined in L^∞ thanks to Theorem 1.1:

$$\left[q\left(u^\tau\right) - q(u_b) - \eta'(u_b)\left(A\left(u^\tau\right) - A(u_b)\right)\right] \cdot \hat{n} \geqslant 0, \tag{7}$$

where B^τ means the trace of B on $\Gamma = (0, T) \times \partial\Omega$ and $\hat{n}$ is the unit outward normal to $\partial\Omega$ with $n = (0, \hat{n})$.

Our second main result is the well-posedness of entropy solutions to the initial–boundary value problem (1), (5), and (6), with the boundary condition (6) being interpreted in the sense of (7).

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a regular open set with $\mathcal{C}^2$ boundary. Assume that the source term $S(t, x, u)$ obeys* (3) *and that the flux function $A \in C^2(\mathbb{R})$ verifies* (2)*. Let $u_0 \in L^\infty(\Omega)$. Then there exists a unique entropy solution $u \in L^\infty(Q)$ verifying* (1)*,* (4)*,* (5)*, and* (7)*.*

This theorem is proved in Section 3. As in [16], the uniqueness argument utilizes the Dubois–LeFloch boundary condition (7) written in a kinetic form. Recently Ammar, Carrillo, and Wittbold [1] showed the well-posedness of conservation laws with source terms by using a more general notion of entropy solutions containing the concept of weak boundary condition introduced by Otto [24]. In contrast to the Kruzhkov approach employed in [1], our proof utilizes a good kinetic formulation of the boundary condition allowing for an adaption of Perthame's "kinetic" uniqueness proof [27,28].

In Section 4 we apply Theorems 1.1 and 1.2 to investigate the well-posedness of the initial–boundary value problem for the so-called *Degasperis–Procesi* equation

$$\partial_t u - \partial^3_{txx} u + 4u\partial_x u = 3\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u, \quad (t, x) \in (0, T) \times (0, 1), \tag{8}$$

augmented with the initial condition

$$u(0, x) = u_0(x), \quad x \in (0, 1), \tag{9}$$

and the boundary data

$$\begin{aligned} u(t,0) &= g_0(t), \qquad u(t,1) = g_1(t), \quad t \in (0,T), \\ \partial_x u(t,0) &= h_0(t), \qquad \partial_x u(t,1) = h_1(t), \quad t \in (0,T). \end{aligned} \tag{10}$$

We assume that

$$\begin{aligned} &u_0 \in L^\infty(0,1), \qquad u_0(0) = g_0(0), \qquad u_0(1) = g_1(0), \\ &g_0, g_1 \in H^1(0,T), \qquad h_0, h_1 \in L^\infty(0,T). \end{aligned} \tag{11}$$

Degasperis and Procesi [10] deduced (8) from the following family of third order dispersive nonlinear equations, indexed over six constants $\alpha, \gamma, c_0, c_1, c_2, c_3 \in \mathbb{R}$:

$$\partial_t u + c_0 \partial_x u + \gamma \partial^3_{xxx} u - \alpha^2 \partial^3_{txx} u = \partial_x \left(c_1 u^2 + c_2 (\partial_x u)^2 + c_3 u \partial^2_{xx} u \right).$$

Using the method of asymptotic integrability, they found that only three equations within this family were asymptotically integrable up to the third order: the *KdV equation* ($\alpha = c_2 = c_3 = 0$), the *Camassa–Holm equation* ($c_1 = -\frac{3c_3}{2\alpha^2}$, $c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}$, $c_2 = c_3$), which properly scaled reads

$$\partial_t u + \partial_x u + 6u \partial_x u + \partial^3_{xxx} u - \alpha^2 \left(\partial^3_{txx} u + \frac{9}{2} \partial_x u \partial^2_{xx} u + \frac{3}{2} u \partial^3_{xxx} u \right) = 0. \tag{12}$$

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, Eq. (12) can be transformed into the form (8), see [8,9] for details.

Degasperis, Holm, and Hone [9] proved the integrability of (8) by constructing a Lax pair. Moreover, they provided a relation to a negative flow in the Kaup–Kupershmidt hierarchy by a reciprocal transformation and derived two infinite sequences of conserved quantities along with a bi-Hamiltonian structure. Furthermore, they showed that the Degasperis–Procesi equation are endowed with weak (continuous) solutions that are superpositions of multipeakons and described the integrable finite-dimensional peakon dynamics. An explicit solution was also found in the perfectly antisymmetric peakon–antipeakon collision case. Lundmark and Szmigielski [21], using an inverse scattering approach, computed n-peakon solutions to (8). Mustafa [23] proved that smooth solutions have infinite speed of propagation, that is, they lose instantly the property of having compact support. Blow-up phenomena have been investigated in, for example, [36]. Regarding the Cauchy problem for the Degasperis–Procesi equation (8), Escher, Liu, and Yin have studied its well-posedness within certain functional classes in a series of papers [12–14,19, 32–35].

The approach taken in the papers just listed emphasizes the similarities between the Degasperis–Procesi equation and the Camassa–Holm equation, and consequently the main focus has been on (weak) continuous solutions. In a rather different direction, Coclite and Karlsen [5–7] and Lundmark [20] initiated a study of discontinuous (shock wave) solutions to the Degasperis–Procesi equation (8). In particular, the existence, uniqueness, and stability of entropy solutions of the Cauchy problem for (8) is proved in [5–7].

When it comes to initial–boundary value problems for the Degasperis–Procesi equation much less is known. The first results in that direction are those of Escher and Yin [14,37], which apply to continuous solutions. To encompass discontinuous solutions we shall herein extend the approach of [5–7], relying on Theorems 1.1 and 1.2 above. Following [5] we rewrite (8), (9), (10) as a hyperbolic–elliptic system with boundary conditions:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x P = 0, & (t,x) \in (0,T) \times (0,1), \\ -\partial_{xx}^2 P + P = \dfrac{3}{2}u^2, & (t,x) \in (0,T) \times (0,1), \\ u(0,x) = u_0(x), & x \in (0,1), \\ u(t,0) = g_0(t), \quad u(t,1) = g_1(t), & t \in (0,T), \\ \partial_x P(t,0) = \psi_0(t), \quad \partial_x P(t,1) = \psi_1(t), & t \in (0,T), \end{cases} \tag{13}$$

where

$$\psi_0 = -g_0' - g_0 h_0, \qquad \psi_1 = -g_1' - g_1 h_1. \tag{14}$$

Let us give a heuristic motivation for the equivalence between (15) and (13). From (8), provided the involved functions are sufficiently smooth,

$$\left(1 - \partial_{xx}^2\right)(\partial_t u + u\partial_x u + \partial_x P) = 0, \tag{15}$$

since, by (14), formally the trace of $\partial_t u + u\partial_x u + \partial_x P$ vanishes at $x = 0$ and $x = 1$, we can invert the differential operator $1 - \partial_{xx}^2$ and pass from (15) to (13).

In the case $g_0 = g_1 = 0$ we do not need any boundary condition on $\partial_x u$, indeed from (14) we have $\psi_0 = \psi_1 = 0$.

The boundary conditions for the P-equation in (13) are of Neumann type. Let $G = G(x, y)$ be the Green's function of the operator $1 - \partial_{xx}^2$ with homogeneous Neumann boundary conditions on $(0, 1)$ and let $Q = Q(t, x)$ be the solution of

$$\begin{cases} -\partial_{xx}^2 Q + Q = 0, \quad (t,x) \in (0,T) \times (0,1), \\ \partial_x Q(t,0) = \psi_0(t), \quad \partial_x Q(t,1) = \psi_1(t), \quad t \in (0,T). \end{cases}$$

The function P has a convolution structure

$$P(t,x) = P^u(t,x) := \frac{3}{2}\int_0^1 G(x,y)u^2(t,y)\,dy + Q(t,x),$$

and (13) can be written as a conservation law with a nonlocal source

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = -\partial_x P^u = -\frac{3}{2}\int_0^1 \partial_x G(x,y)u^2(t,y)\,dy - \partial_x Q(t,x). \tag{16}$$

Due to the regularizing effect of the elliptic equation in (13) we have that

$$u \in L^\infty\big((0,T) \times (0,1)\big) \quad \Rightarrow \quad P^u \in L^\infty\big(0,T; W^{2,\infty}(0,1)\big). \tag{17}$$

Therefore, if a map $u \in L^\infty((0,T) \times (0,1))$ satisfies, for every convex map $\eta \in C^2$,

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \leqslant 0, \quad q(u) = \int^u \xi \eta'(\xi)\, d\xi, \tag{18}$$

in the sense of distributions, then Theorem 1.1 provides the existence of strong traces u_0^τ, u_1^τ on the boundaries $x = 0, 1$, respectively.

We say that $u \in L^\infty((0,T) \times (0,1))$ is an entropy solution of the initial–boundary value problem (8), (9), (10) if

(i) u is a distributional solution of (13);
(ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality (18) holds in the sense of distributions;
(iii) for every convex function $\eta \in C^2$ with corresponding q defined by $q'(u) = u\eta'(u)$, the boundary entropy condition

$$\begin{aligned} & q\big(u_0^\tau(t)\big) - q\big(g_0(t)\big) - \eta'\big(g_0(t)\big)\frac{(u_0^\tau(t))^2 - (g_0(t))^2}{2} \\ & \quad \leqslant 0 \leqslant q\big(u_1^\tau(t)\big) - q\big(g_1(t)\big) - \eta'\big(g_1(t)\big)\frac{(u_1^\tau(t))^2 - (g_1(t))^2}{2} \end{aligned} \tag{19}$$

holds for a.e. $t \in (0,T)$.

Our main result for the initial–boundary value problem for the Degasperis–Procesi equation is the following theorem, which is proved in Section 4.

Theorem 1.3. *Let u_0, γ, g_0, g_1, h_0, h_1 satisfy* (11). *The initial–boundary value problem* (8), (9), (10) *possesses a unique entropy solution $u \in L^\infty((0,T) \times (0,1))$.*

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1, adapting Vasseur's blow-up method [31].

2.1. Weak boundary trace

We first reformulate the relevant problems on local open subsets and construct weak boundary traces of entropy solutions on these local sets. The reason for working on local subsets is that we are going to use the blow-up method. We split the boundary into a countable number of subsets. Indeed, for each $\hat{x} \in \partial\Omega$, there exist $r_{\hat{x}} > 0$, a C^2 mapping $\widetilde{\gamma}_{\hat{x}} : \mathbb{R}^{d-1} \to \mathbb{R}$, and an isometry for the Euclidean norm $\mathcal{R}_{\hat{x}} : \mathbb{R}^d \to \mathbb{R}^d$ such that, upon rotating, relabeling, and translating the coordinate axes if necessary,

$$\begin{gathered} \mathcal{R}_{\hat{x}}(\hat{x}) = 0, \\ \mathcal{R}_{\hat{x}}(\Omega) \cap (-r_{\hat{x}}, r_{\hat{x}})^d = \big\{ y = (y_0, \hat{y}) \in (-r_{\hat{x}}, r_{\hat{x}})^d \;\big|\; y_0 > \widetilde{\gamma}_{\hat{x}}(\hat{y}) \big\}. \end{gathered}$$

We have

$$\partial\Omega \subset \bigcup_{\hat{x} \in \partial\Omega} \mathcal{R}_{\hat{x}}^{-1}\big((-r_{\hat{x}}, r_{\hat{x}})^d\big).$$

Hence, for each $\hat{z} = (\hat{t}, \hat{x}) \in \Gamma$, we obtain an isometry map $\Lambda_{\hat{z}} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ given by $\Lambda_{\hat{z}}(t, x) = (y_0, t - \hat{t}, \hat{y})$, where $(y_0, \hat{y}) = \mathcal{R}_{\hat{x}}(x)$. Then we have

$$\Gamma = \bigcup_{\hat{z} \in \Gamma} (\Lambda_{\hat{z}})^{-1}(\Gamma_{\hat{z}}),$$

where

$$\Gamma_{\hat{z}} = \{(y_0, t - \hat{t}, y) \mid y_0 = \tilde{\gamma}_{\hat{x}}(\hat{y})\}.$$

Let us denote $w = (w_0, \hat{w}) := (y_0, t - \hat{t}, \hat{y})$ and $\hat{w} := (t - \hat{t}, \hat{y})$. Since the above collection of open sets is countable,

$$\bigcup_{\hat{z} \in \Gamma} (\Lambda_{\hat{z}})^{-1}(\Gamma_{\hat{z}}) = \bigcup_{\alpha \in K} (\Lambda_\alpha)^{-1}(\Gamma_\alpha),$$

where K is a countable set and

$$\Gamma_\alpha = \Lambda_\alpha^{-1}\big(\{w \in (-r_\alpha, r_\alpha)^{d+1} \mid w_0 = \gamma_\alpha(\hat{w})\}\big),$$

where $\gamma_\alpha(\hat{w}) := \tilde{\gamma}_\alpha(\hat{y})$. We define

$$Q_\alpha = \{w \in (-r_\alpha, r_\alpha)^{d+1} \mid w_0 > \gamma_\alpha(\hat{w})\}.$$

In an attempt to simplify the notation we write α instead of $\hat{z}_\alpha$ in the indices. From now on we will work in Q_α and state the equations in terms of the new w variable. To this end, define $u_\alpha : Q_\alpha \to \mathbb{R}$ by $u_\alpha(w) = u((\Lambda_\alpha)^{-1}(w))$ and set $A_\alpha(\xi) = \Lambda_\alpha(\xi, A(\xi))$, $q_\alpha(\xi) = \Lambda_\alpha(\eta(\xi), q(\xi))$. For every fixed α, every deformation ψ, and every $\hat{w} \in (-r_\alpha, r_\alpha)^d$, we define

$$\tilde{\psi}(s, \hat{w}) = (\Lambda_\alpha \circ \psi)\big(s, (\Lambda_\alpha)^{-1}\big(\gamma_\alpha(\hat{w}), \hat{w}\big)\big), \quad s \in [0, 1]. \tag{20}$$

In terms of the w variable, (1) and (4) read respectively

$$\operatorname{div}_w A_\alpha(u_\alpha) = S(w, u_\alpha) \quad \text{in } Q_\alpha \tag{21}$$

and

$$\operatorname{div}_w q_\alpha(u_\alpha) \leqslant \eta'(u_\alpha) S(w, u_\alpha) \quad \text{in } Q_\alpha. \tag{22}$$

We now introduce a kinetic formulation of (21) and (22), cf. [18]. To do so we set $L = \|u\|_{L^\infty(\Omega)}$, bring in a new variable $\xi \in (-L, L)$, and introduce for every $v \in (-L, L)$ the function

$$\chi(v, \xi) = \begin{cases} \mathbf{1}_{\{0 \leqslant \xi \leqslant v\}}, & \text{if } v \geqslant 0, \\ -\mathbf{1}_{\{v \leqslant \xi \leqslant 0\}}, & \text{if } v < 0. \end{cases}$$

To effectively represent weak limits of nonlinear functions of weakly converging sequences, we introduce new functions, called microscopic functions, which depend on ξ and on an additional variable z [28].

Definition 2.1. *Let N be an integer and $\mathcal{O}$ be an open set of $\mathbb{R}^N$. We say that $f \in L^\infty(\mathcal{O} \times (-L, L))$ is a microscopic function if it obeys $0 \leqslant \operatorname{sgn}(\xi) f(z,\xi) \leqslant 1$ for almost every (z,ξ). We say that f is a χ-function if there exists a function $u \in L^\infty(\mathcal{O})$ such that for a.e. $z \in \mathcal{O}$ there holds $f(z,\cdot) = \chi(u(z),\cdot)$.*

For later use, let us collect the following results (cf. [28]).

Lemma 2.1. *Fix an open set $\mathcal{O} \subset \mathbb{R}^N$, and let $f_k \in L^\infty(\mathcal{O} \times (-L, L))$ be a sequence of χ-functions $L^\infty_{\text{weak-}\star}$-converging to $f \in L^\infty(\mathcal{O} \times (-L, L))$. Introduce the functions $u_k(\cdot) = \int_{-L}^{L} f_k(\cdot,\xi)\,d\xi$ and $u(\cdot) = \int_{-L}^{L} f(\cdot,\xi)\,d\xi$. Then, for almost every $z \in \mathcal{O}$, the function $f(z,\cdot)$ lies in $BV(-L, L)$. Moreover, the following statements are equivalent*:

- *f_k converges strongly to f in $L^1_{\mathrm{loc}}(\mathcal{O} \times (-L, L))$.*
- *u_k converges strongly to u in $L^1_{\mathrm{loc}}(\mathcal{O})$.*
- *f is a χ-function.*

Observe that if f is a χ-function then $u(z) = \int_{-L}^{L} f(z,\xi)\,d\xi$. The following theorem is due to Lions, Perthame, and Tadmor [18].

Theorem 2.1. *A function $u \in L^\infty(Q_\alpha)$, with $|u| \leqslant L$, is a solution of (21) and (22) if and only if there exists a non-negative measure $m \in \mathcal{M}^+(Q_\alpha \times (-L, L))$ such that the related χ-function f defined by $f(u(w),\xi) = \chi(u(w),\xi)$ for almost every $(w,\xi) \in (Q_\alpha \times (-L, L))$ verifies*

$$a(\xi) \cdot \nabla_w f + S(\cdot,\xi)\big(\partial_\xi f - \delta(\xi)\big) = \partial_\xi m \quad \text{in } \mathcal{D}'\big(Q_\alpha \times (-L, L)\big), \tag{23}$$

where $a(\xi) := A'_\alpha(\xi)$.

Denote a by $a = (a_0, \hat{a})$. To simplify the notation we keep denoting the normal vectors by n_s and n.

In what follows, for each fixed α, we will consider the set Q_α and the χ-function f associated to u_α. For any regular deformation ψ and $\hat{w} \in (-r_\alpha, r_\alpha)^d$ we set

$$f_\psi(s,\hat{w},\xi) = f\big(\tilde{\psi}(s,\hat{w}),\xi\big),$$

where $\tilde{\psi}$ is defined in (20). We now show that f_ψ has a weak trace at $s = 0$, which does not depend on the deformation ψ, i.e., the way chosen to reach the boundary.

Lemma 2.2. *Let f be a solution of (23) in $Q_\alpha \times (-L, L)$. Then there exists*

$$f^\tau \in L^\infty\big((-r_\alpha, r_\alpha)^d \times (-L, L)\big)$$

such that

$$\operatorname*{ess\,lim}_{s\to 0} f_\psi(s,\cdot,\cdot) = f^\tau \quad \text{in } H^{-1}\big((-r_\alpha, r_\alpha)^d \times (-L, L)\big),$$

for all Γ_α-regular deformation ψ. Moreover, f^τ is uniquely defined.

Proof. Since $\|f_\psi(s,\cdot,\cdot)\|_{L^\infty} \leqslant 1$, by weak compactness and the Sobolev imbedding theorem, for every sequence $s^k \xrightarrow{k\to\infty} 0$ there exist a subsequence $k_p \xrightarrow{p\to\infty} \infty$ and a function $g_\psi^\tau \in L^\infty((-r_\alpha, r_\alpha)^d \times (-L, L))$ such that

$$f_\psi\bigl(s^{k_p},\cdot,\cdot\bigr) \xrightarrow{p\to\infty} g_\psi^\tau \quad \text{in } H^{-1} \cap L^\infty_{\text{weak-}\star}, \tag{24}$$

for every regular deformation ψ. Let us show that g_ψ^τ is independent of the deformation ψ and the sequence s^k and its subsequence s^{k_p}. To do so, let us first consider the entropy flux

$$\bar{q}_\eta(w) = \int_{-L}^{L} a(\xi)\eta'(\xi) f(w,\xi)\, d\xi, \tag{25}$$

associated with the entropy η. Multiplying (23) by $\eta'(\xi)$ and integrating it with respect to ξ we find

$$\operatorname{div}_w \bar{q}_\eta = -\int_{-L}^{L} \bigl[\eta''(\xi)m_1 - \eta'(\xi)m_2\bigr](w, d\xi) \in \mathcal{M}\bigl((-r_\alpha, r_\alpha)^{d+1}\bigr),$$

where

$$m_1 = -Sf + m, \qquad m_2 = \partial_\xi Sf + \delta(\xi)S. \tag{26}$$

We can now use the following theorem (cf. Chen and Frid [3]):

Theorem 2.2. *Let Ω be an open set with regular boundary $\partial\Omega$ and $F \in [L^\infty(\Omega)]^{d+1}$ be such that* $\operatorname{div}_y F$ *is a bounded measure. Then there exists $F\cdot n \in L^\infty(\partial\Omega)$ such that for every $\partial\Omega$-regular deformation ψ,*

$$\operatorname*{ess\,lim}_{s\to 0} F\bigl(\psi(s,\cdot)\bigr)\cdot n_s(\cdot) = F\cdot n \quad \textit{in } L^\infty_{\text{weak-}\star}(\partial\Omega),$$

where n_s is a unit outward normal field of $\psi(\{s\}\times\partial\Omega)$.

This theorem ensures the existence of a function $\bar{q}_\eta^\tau \cdot n \in L^\infty((-r_\alpha, r_\alpha)^d)$, which does not depend on ψ, such that

$$\bar{q}_\eta\bigl(\tilde{\psi}(s,\cdot)\bigr)\cdot n_s(\cdot) \xrightarrow{s\to 0} \bar{q}_\eta^\tau \cdot n \quad \text{in } \mathcal{D}'\bigl((-r_\alpha, r_\alpha)^d\bigr), \tag{27}$$

for every regular deformation ψ. The function n_s converges strongly to n, i.e., the unit outward normal to Q_α. The convergence takes place in $L^1((-r_\alpha, r_\alpha)^d)$. So, using (25) and (24), (27), we obtain

$$\int_{(-r_\alpha,r_\alpha)^d} \int_{-L}^{L} \varphi(\hat{w})\eta'(\xi)a(\xi)\cdot n(\hat{w}) g_\psi^\tau(\hat{w},\xi)\, d\xi\, d\hat{w} = \int_{(-r_\alpha,r_\alpha)^d} \bar{q}_\eta^\tau \cdot n(\hat{w})\varphi(\hat{w})\, d\hat{w},$$

for every test functions $\varphi \in \mathcal{D}((-r_\alpha, r_\alpha)^d)$. The right-hand side of this equation is independent of ψ, the sequence s^k and its subsequence s^{k_p}, so g_ψ^τ does not depend on those quantities either thanks to (2). The result is obtained from the uniqueness of the limit. $\square$

2.2. Strong boundary trace

Let us now show that entropy solutions possess a strong boundary trace. To do so we will employ the blow-up method [31] and apply the averaging lemma to conclude that $f^\tau(\hat{w}, \cdot)$ is a χ-function for almost every $(\hat{w}, \xi) \in (-r_\alpha, r_\alpha)^d \times (-L, L)$. To this end, we shall rely on the following lemma, which is a straightforward consequence of Lemma 2.1.

Lemma 2.3. *The function f^τ is a χ-function if and only if*

$$\operatorname*{ess\,lim}_{s \to 0} f_\psi(s, \cdot, \cdot) = f^\tau \quad \text{in } L^1\big((-r_\alpha, r_\alpha)^d\big),$$

for any deformation ψ.

Let fix a specific deformation on Q_α, namely

$$\tilde{\psi}_0(s, \hat{w}) = \big(s + \gamma_\alpha(\hat{w}), \hat{w}\big). \tag{28}$$

We use the notation

$$\tilde{f}(s, \hat{w}, \xi) = f_{\tilde{\psi}_0}(s, \hat{w}, \xi) = f\big(\tilde{\psi}_0(s, \hat{w}), \xi\big),$$

when we work with the deformation (28). Indeed, it is enough to show the strong trace of f_ψ for the specific deformation (28) thanks to Lemma 2.2.

Notice that $\tilde{\psi}_0(s, \hat{w}) \in Q_\alpha$ if and only if $\hat{w} \in (-r_\alpha, r_\alpha)^d$ and $0 < s < r_\alpha$. From (23) we find that $\tilde{f}$ is a solution of

$$\tilde{a}^0(\hat{w}, \xi)\partial_s \tilde{f} + \hat{a}(\xi) \cdot \nabla_{\hat{w}} \tilde{f} = \partial_\xi \tilde{m}_1 + \tilde{m}_2, \tag{29}$$

where $\tilde{m}_i(s, \hat{w}, \xi) = m_i(\tilde{\psi}_0(s, \hat{w}), \xi)$ with m_i defined in (26), $i = 1, 2$, and $\tilde{a}^0(\hat{w}, \xi) = \lambda(\hat{w}) a(\xi) \cdot n(\hat{w})$, where

$$\lambda(\hat{w}) := -\sqrt{1 + \big|\nabla \gamma_\alpha(\hat{w})\big|^2}.$$

Before introducing the notion of rescaled solution, let us state two lemmas (cf. [31]).

Lemma 2.4. *There exist a sequence δ_k which converges to 0 and a set $\mathcal{E} \subset (-r_\alpha, r_\alpha)^d$ with $\mathcal{L}((-r_\alpha, r_\alpha)^d \setminus \mathcal{E}) = 0$ such that for every $\hat{w} \in \mathcal{E}$ and every $R > 0$,*

$$\lim_{k \to \infty} \frac{1}{\delta_n^d} |\tilde{m}_i|\big((0, R\delta_k) \times \big(\hat{w} + (-R\delta_k, R\delta_k)^d\big) \times (-L, L)\big) = 0, \quad i = 1, 2.$$

Lemma 2.5. *There exist a subsequence, still denoted by δ_k, and a subset $\mathcal{E}'$ of $(-r_\alpha, r_\alpha)^d$ with $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{L}((-r_\alpha, r_\alpha)^d \setminus \mathcal{E}') = 0$, such that for every $\hat{w} \in \mathcal{E}'$ and every $R > 0$ there holds*

$$\lim_{\delta_k \to 0} \int_{-L}^{L} \int_{(-R,R)^d} \left| f^\tau(\hat{w}, \xi) - f^\tau(\hat{w} + \delta_k \underline{\hat{y}}, \xi) \right| d\underline{\hat{y}}\, d\xi = 0,$$

$$\lim_{\delta_k \to 0} \int_{-L}^{L} \int_{(-R,R)^d} \left| \tilde{a}^0(\hat{w}, \xi) - \tilde{a}^0(\hat{w} + \delta_k \underline{\hat{y}}, \xi) \right| d\underline{\hat{y}}\, d\xi = 0.$$

Let us now introduce the localization method [31]. We use the notation

$$Q_\alpha^\delta = (0, r_\alpha/\delta) \times (-r_\alpha/\delta, r_\alpha/\delta)^d.$$

The goal is to show that for every $\hat{w} \in \mathcal{E}'$, $f^\tau(\hat{w}, \cdot)$ is a χ-function. From now on we fix such a $\hat{w} \in \mathcal{E}'$. Then we rescale the $\tilde{f}$ function by introducing a new function $\tilde{f}_\delta$, which depends on new variables $(\underline{s}, \underline{\hat{y}}) \in Q_\alpha^\delta$, defined by

$$\tilde{f}_\delta(\underline{s}, \underline{\hat{y}}, \xi) = \tilde{f}(\delta \underline{s}, \hat{w} + \delta \underline{\hat{y}}, \xi).$$

This function depends obviously on $\hat{w}$ but since it is fixed throughout this section, we skip it in the notation. The function $\tilde{f}_\delta$ is still a χ-function and we notice that

$$\tilde{f}_\delta(0, \underline{\hat{y}}, \xi) = f^\tau(\hat{w} + \delta \underline{\hat{y}}, \xi).$$

Hence we gain knowledge about $f^\tau(\hat{w}, \cdot)$ by studying the limit of $\tilde{f}_\delta$ when $\delta \to 0$. We define

$$\tilde{a}_\delta^0(\underline{\hat{y}}, \xi) = \tilde{a}^0(\hat{w} + \delta \underline{\hat{y}}, \xi).$$

In view of (29),

$$\tilde{a}_\delta^0(\underline{\hat{y}}, \xi)\partial_{\underline{s}} \tilde{f}_\delta + \hat{a}(\xi) \cdot \nabla_{\underline{\hat{y}}} \tilde{f}_\delta = \partial_\xi \tilde{m}_\delta^1 + \tilde{m}_\delta^2, \tag{30}$$

where $\tilde{m}_\delta^i$ is the non-negative measure defined for every real numbers $R_1^j < R_2^j$, $L_1 < L_2$ by

$$\tilde{m}_\delta^i\left(\prod_{0 \leqslant j \leqslant d} \left[R_1^j, R_2^j\right] \times [L_1, L_2]\right) = \frac{1}{\delta^d} \tilde{m}_i\left(\prod_{0 \leqslant j \leqslant d} \left[\hat{w}_j + \delta R_1^j, \hat{w}_j + \delta R_2^j\right] \times [L_1, L_2]\right),$$

for $i = 1, 2$.

We now pass to the limit when δ goes to 0 in the rescaled equation. To this end, we shall need to prove strong convergence via an application of an averaging lemma taken from Perthame and Souganidis [29].

Lemma 2.6. *Let N be an integer, f_n bounded in $L^\infty(\mathbb{R}^{N+1})$ and $\{h_n^1, h_n^2\}$ be relatively compact in $[L^p(\mathbb{R}^{N+1})]^{2N}$ with $1 < p < +\infty$ solutions of the transport equation*:

$$a(\xi)\cdot\nabla_y f_k = \partial_\xi\big(\nabla_y\cdot h_k^1\big) + \nabla_y\cdot h_k^2,$$

where $a \in [C^2(\mathbb{R})]^N$ verifies the non-degeneracy condition (2). *Let $\phi \in \mathcal{D}(\mathbb{R})$, then the average $u_k^\phi(w) = \int_{\mathbb{R}} \phi(\xi) f_k(w,\xi)\,d\xi$ is relatively compact in $L^p(\mathbb{R}^N)$.*

Lemma 2.7. *There exist a sequence $\delta_k \to 0$ and a χ-function $\tilde f_\infty \in L^\infty(\mathbb{R}^+\times\mathbb{R}\times(-L,L))$ such that $\tilde f_{\delta_n}$ converges strongly to $\tilde f_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}\times(-L,L))$ and*

$$\tilde a^0(\hat w,\xi)\partial_{\underline{s}}\tilde f_\infty + \hat a(\xi)\cdot\partial_{\underline{\hat w}}\tilde f_\infty = 0. \tag{31}$$

Proof. We consider the sequence δ_n of Lemma 2.5. By weak compactness, there exists a function $\tilde f_\infty \in L^\infty(\mathbb{R}^+\times\mathbb{R}^d\times(-L,L))$ such that, up to extraction of a subsequence, $\tilde f_{\delta_n}$ converges to $\tilde f_\infty$ in $L^\infty_{\text{weak-}\star}$. Thanks to Lemma 2.4, $\tilde m^i_{\delta_n}$ converges to 0 in the sense of measures. So passing to the limit in (30) gives (31).

First, we localize in $(\underline{w},\xi)$. For any $R > 0$ big enough, we consider Φ_1, Φ_2 with values in $[0,1]$ such that $\Phi_1 \in \mathcal{D}(\mathbb{R}^+\times\mathbb{R}^d)$, $\Phi_2 \in \mathcal{D}(\mathbb{R})$, and $\mathrm{Supp}(\Phi_1) \subset (1/(2R), 2R)\times(-2R,2R)^d$, $\mathrm{Supp}(\Phi_2) \subset (-2L, 2L)$. Moreover, $\Phi_1(\underline{w}) = 1$ for $\underline{w} \in (1/R, R)\times(-R,R)^d$ and $\Phi_2(\xi) = 1$ for $\xi \in (-L,L)$. Hence for $\delta < r_\alpha/(2R)$, we can define on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ the function

$$\tilde f_\delta^R = \Phi_1\Phi_2\tilde f_\delta,$$

(where $\tilde f_\delta^R = 0$ if $\tilde f_\delta$ is not defined). On $(1/R,R)\times(-R,R)\times(-L,L)$ we have $\tilde f_\delta^R = \tilde f_\delta$. So, if we denote by $a_{\hat w}(\xi) = (\tilde a^0(\hat w,\xi), \hat a(\xi))$ (which depends only on ξ since $\hat w$ is fixed), from (30) we get

$$\begin{aligned}
a_{\hat w}(\xi)\cdot\nabla_{\underline{\hat y}}\tilde f_\delta^R &= \partial_\xi\big(\Phi_1\Phi_2\tilde m_\delta^1\big) - \Phi_1\Phi_2'\tilde m_\delta^1 + a_{\hat w}(\xi)\cdot\nabla_{\underline{\hat y}}\Phi_1\Phi_2\tilde f_\delta^R + \Phi_1\Phi_2\tilde m_\delta^2\\
&\quad + \partial_s\big[\big(\tilde a^0(\hat w,\xi) - \tilde a^0_\delta(\underline{\hat y},\xi)\big)\tilde f_\delta^R\big]\\
&= \partial_\xi\mu_{1,\delta} + \mu_{2,\delta} + \partial_s\big[\big(\tilde a^0(\hat w,\xi) - \tilde a^0_\delta(\underline{\hat y},\xi)\big)\tilde f_\delta^R\big],
\end{aligned}$$

where μ_{1,δ_k} and μ_{2,δ_k} are measures uniformly bounded with respect to k. In view of Lemma 2.5 we can see that $\tilde a^0(\hat w,\xi) - \tilde a^0_\delta(\underline{\hat y},\xi)$ converges to 0 in $L^1_{\mathrm{loc}}(\mathbb{R}^d\times(-L,L))$. So it converges to 0 in L^p_{loc} for every $1 \leqslant p < \infty$ since these functions are bounded in L^∞. Since the measures are compactly imbedded in $W^{-1,p}$ for $1 \leqslant p < \frac{d+2}{d+1}$, we can apply Lemma 2.6 with $N = d+1$, $f_k = \tilde f^R_{\delta_k}$, $\phi(\xi) = \Phi_2(\xi)$, and $a(\xi) = a_{\hat w}(\xi)$. It follows that $\int \tilde f_\delta^R\Phi_2(\xi)\,d\xi$ is compact in L^p for $1 \leqslant p < \frac{d+2}{d+1}$. So by uniqueness of the limit, $\int \tilde f_{\delta_n}(\cdot,\xi)\,d\xi$ converges strongly to $\int \tilde f_\infty(\cdot,\xi)\,d\xi$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$. Lemma 2.1 ensures us that $\tilde f_{\delta_n}$ converges strongly to $\tilde f_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}\times(-L,L))$ and moreover that $\tilde f_\infty$ is a χ-function. $\square$

We now turn to the characterization of the limit function $\tilde f_\infty$.

Lemma 2.8. *(See [31].) For every $\hat{w} \in \mathcal{E}'$, $\tilde{f}_\infty(\underline{w}, \xi) = f^\tau(\hat{w}, \xi)$ for almost every $(\underline{w}, \xi) \in \mathbb{R}^{d+1} \times (-L, L)$, and the function $f^\tau(\hat{w}, \cdot)$ is a χ-function.*

Thus, from Propositions 2.3 and 2.8, we can prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] For every α and every deformation ψ, we have

$$\operatorname*{ess\,lim}_{s\to 0} \int\limits_{(-r_\alpha, r_\alpha)^d} \int\limits_{-L}^{L} \left| f_\psi(s, \hat{w}, \xi) - f^\tau(\hat{w}, \xi) \right| d\xi \, d\hat{w} = 0.$$

We define u^τ by

$$u^\tau(\hat{z}) = \int\limits_{-L}^{L} f^\tau(\hat{w}, \xi)\, d\xi, \quad \text{if } \big(\gamma_\alpha(\hat{w}), \hat{w}\big) = \Lambda_\alpha(\hat{z}).$$

For every compact subset K of $(0, T) \times \partial\Omega$, there exists a finite set I_0 such that $K \subset \bigcup_{\alpha \in I_0}$ and

$$\int\limits_K \left| u\big(\psi(s, \hat{z})\big) - u^\tau(\hat{z}) \right| d\sigma(\hat{z}) \leqslant \sum_{\alpha \in I_0} \int\limits_{\Gamma_\alpha} \left| u\big(\psi(s, \hat{z})\big) - u^\tau(\hat{z}) \right| d\sigma(\hat{z}),$$

which converges to 0 as s tends to 0. This concludes the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

3.1. Existence proof

In this section we will show the existence of an entropy solution for the initial–boundary value problem (1), (5), and (6), with the boundary condition (6) interpreted in the sense of (7).

Let $\{S^\varepsilon\}_{\varepsilon>0}$ be a sequence of smooth functions converging in L^1_{loc} to S with respect to variables (t, x), for example obtained by mollifying the function S, and consider smooth solutions to the uniformly parabolic equation

$$\partial_t u^\varepsilon + \operatorname{div}_x A\big(u^\varepsilon\big) = S^\varepsilon\big(t, x, u^\varepsilon\big) + \varepsilon \Delta_x u^\varepsilon, \tag{32}$$

with initial and boundary data

$$u^\varepsilon(0, \cdot) = u_0 \qquad u^\varepsilon\big|_\Gamma = u_b. \tag{33}$$

For the sake of simplicity in this proof, we will assume that the data u_0, u_b are smooth functions. Then, for each $\varepsilon > 0$, the existence of a unique smooth solution of the initial–boundary (32), (33) value problem is a standard result.

By the maximum principle,

$$\left| u^\varepsilon(t, x) \right| \leqslant \|u_0\|_{L^\infty} + \|u_b\|_{L^\infty} + \|S\|_{L^\infty} T. \tag{34}$$

For any convex entropy function η and corresponding entropy flux function q with $q' = \eta' A'$, multiplying (32) $\eta'(u^\varepsilon)$ yields

$$\partial_t \eta\big(u^\varepsilon\big) + \operatorname{div}_x q\big(u^\varepsilon\big) - \eta'\big(u^\varepsilon\big) S^\varepsilon\big(t,x,u^\varepsilon\big) = \varepsilon \Delta_x \eta\big(u^\varepsilon\big) - \varepsilon \eta''\big(u^\varepsilon\big)\big|\nabla_x u^\varepsilon\big|^2. \tag{35}$$

For any function $\varphi \in C_c^\infty(Q)$, it follows from (35) that

$$\begin{aligned}\int_Q \eta\big(u^\varepsilon\big)\partial_t\varphi + q\big(u^\varepsilon\big)\cdot\nabla_x\varphi\,dt\,dx = \int_Q \varepsilon\eta'\big(u^\varepsilon\big)\nabla_x u^\varepsilon\cdot\nabla_x\varphi\,dt\,dx + \int_Q \eta''\big(u^\varepsilon\big)\varepsilon\big|\nabla_x u^\varepsilon\big|^2\varphi\,dt\,dx \\ - \int_Q S^\varepsilon\big(t,x,u^\varepsilon\big)\eta'\big(u^\varepsilon\big)\varphi\,dt\,dx.\end{aligned} \tag{36}$$

Let K be an arbitrary compact subset of Q and choose in (36) a function $\varphi \in C_c^\infty(Q)$ satisfying

$$\varphi|_K = 1, \quad 0 \leqslant \varphi \leqslant 1.$$

It follows that

$$\int_Q \big|S^\varepsilon\big(t,x,u^\varepsilon\big)u^\varepsilon\varphi\big|\,dt\,dx \leqslant C\big(T,\varphi,\|u_0\|_{L^\infty}\big)\|S\|_{L^\infty},$$

thanks to (34). Consequently,

$$\int_Q \varepsilon\big|\nabla_x u^\varepsilon\big|^2\,dt\,dx \leqslant C. \tag{37}$$

We can now make obvious the strong convergence of solutions u^ε to (1). Set $f^\varepsilon(t,x,\xi) = \chi(u^\varepsilon,\xi)$, where u^ε is a solution of Eq. (1). Then

$$\begin{aligned}&\partial_t f^\varepsilon + A'(\xi)\cdot\nabla_x f^\varepsilon \\ &\quad = \sum_{j=1}^d \partial_{x_j}\big(\varepsilon\partial_{x_j}u^\varepsilon\delta\big(u^\varepsilon-\xi\big)\big) + \partial_\xi\big(\varepsilon\big|\nabla_x u^\varepsilon\big|^2\delta\big(u^\varepsilon-\xi\big)\big) + S^\varepsilon(\cdot,\cdot,\xi)\delta\big(u^\varepsilon-\xi\big) \\ &\quad =: \sum_{j=1}^d \partial_{x_j}\Gamma_j^\varepsilon + \partial_\xi\Lambda_1^\varepsilon + \Lambda_2^\varepsilon \quad \text{in } \mathcal{D}'.\end{aligned} \tag{38}$$

For any $\sigma \in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})$,

$$\big|\big\langle\Lambda_2^\epsilon,\sigma\big\rangle\big| \leqslant \|\sigma\|_{L^\infty}\int_{\operatorname{supp}(\sigma)}\big|S^\varepsilon\big(t,x,u^\varepsilon\big)\big|\,dt\,dx \leqslant C(T,\sigma). \tag{39}$$

Thus, Λ_2^ϵ is a bounded measure thanks to (39). Now, we may use the Sobolev injection to represent

$$\Lambda_2^\varepsilon(t,x,\xi) = \operatorname{div}_{(t,x,\xi)} \lambda_2^\varepsilon(t,x,\xi), \tag{40}$$

where $\lambda_2^\epsilon(t,x,\xi)$ is compact in $L^q(\mathbb{R}^{d+2})$ for some $q > 1$. Combining (40) and Lemma 3.1 below yields

$$\partial_t f^\varepsilon + A'(\xi)\cdot\nabla_x f^\varepsilon = \sum_{j=1}^d \partial_{x_j}\big(\overline{\gamma_j}^\varepsilon + \partial_\xi \gamma_j^\varepsilon\big) + \partial_\xi \operatorname{div}_{(t,x,\xi)} \lambda_1^\varepsilon + \operatorname{div}_{(t,x,\xi)} \lambda_2^\varepsilon, \tag{41}$$

where $\overline{\gamma_j}^\varepsilon, \gamma_j^\varepsilon \to 0$ in L^2 for $j = 1,\dots,d$ and λ_i^ε is bounded in $W^{1,q}_{\mathrm{loc}}$ for $i = 1, 2$.

Lemma 3.1. *(See [15].) Consider Γ_j^ε and Λ_1^ε given in* (38). *Then*

$$\Gamma_j^\varepsilon = \overline{\gamma_j}^\varepsilon + \partial_\xi \gamma_j^\varepsilon \quad \textit{and} \quad \Lambda_1^\varepsilon = \operatorname{div}_{(t,x,\xi)} \lambda_1^\varepsilon,$$

where $\overline{\gamma_j}^\varepsilon, \gamma_j^\varepsilon \to 0$ in L^2 for $j = 1,\dots,d$ and λ_i^ε is bounded in $W^{1,q}_{\mathrm{loc}}$, $i = 1, 2$.

By (34), (37), and (38), Lemma 2.6 applied to (41) shows that f^ε converges strongly to f in $L^p(Q\times\mathbb{R})$ for some $p > 1$. Let us set $u = \int_{\mathbb{R}} f\,d\xi$. We now conclude the existence of subsequence, still labeled $u^\varepsilon := \int_{\mathbb{R}} f^\varepsilon\,d\xi$, converging to a limit u a.e. and in L^1_{loc} such that the interior entropy inequality holds:

$$\int_Q \eta(u)\partial_t\phi + q(u)\cdot\nabla_x\phi + \eta'(u)S(t,x,u)\phi\,dt\,dx \geqslant 0, \quad \forall\phi \in C_c^\infty(Q),\ \phi \geqslant 0.$$

It remains to prove that the Dubois–LeFloch boundary condition (7) is satisfied.

Lemma 3.2. *Let u be the limit function constructed above. Then, for any convex entropy–entropy flux pair (η, q),*

$$\big[q\big(u^\tau\big) - q(u_b) - \eta'(u_b)\big(A\big(u^\tau\big) - A(u_b)\big)\big]\cdot\hat{n} \geqslant 0$$

where u^τ is the trace of u on $(0,T)\times\partial\Omega$ and $\hat{n}$ is the unit outward normal to $\partial\Omega$.

Proof. We need a family of boundary layer functions $\{\zeta_\delta\} \in C^\infty(\Omega;[0,1])$ verifying

$$\zeta_\delta|_{\Omega_\delta} = 0, \qquad \zeta_\delta|_{\partial\Omega} = 1, \quad \text{and} \quad |\nabla\zeta| \leqslant \frac{c}{\delta^d},$$

where $\Omega_\delta = \{x\in\Omega \mid \operatorname{diam}(x,\partial\Omega) > \delta\}$ and c is a constant independent of δ. Multiplying (35) by $\theta(t,x)\zeta_\delta(x)$ with $\theta \in C_c^\infty(\mathbb{R}^{d+1})$, $\theta \geqslant 0$, we obtain $E_1 = E_2$, where the terms E_1, E_2 are defined and analyzed below.

Integration by parts yields

$$\begin{aligned}
E_1 \ &:= \int_Q \big(\partial_t \eta\big(u^\varepsilon\big) + \operatorname{div}_x q\big(u^\varepsilon\big) - \eta'\big(u^\varepsilon\big) S^\varepsilon\big(t,x,u^\varepsilon\big)\big)\theta(t,x)\zeta_\delta(x)\,dt\,dx \\
&= -\int_Q \eta\big(u^\varepsilon\big)\partial_t\theta(t,x)\zeta_\delta(x) + q\big(u^\varepsilon\big)\cdot\nabla_x\zeta_\delta(x)\theta(t,x) + q\big(u^\varepsilon\big)\cdot\nabla_x\theta(t,x)\zeta_\delta(x) \\
&\quad + \eta'\big(u^\varepsilon\big)S^\varepsilon\big(t,x,u^\varepsilon\big)\theta(t,x)\zeta_\delta(x)\,dt\,dx + \int_{(0,T)\times\partial\Omega} q(u_b)\cdot\hat{n}\,\theta(t,\hat{x})\,dt\,d\sigma \\
&\xrightarrow{\varepsilon\to 0} -\int_Q \eta(u)\partial_t\theta(t,x)\zeta_\delta(x) + q(u)\cdot\nabla_x\zeta_\delta(x)\theta(t,x) + q(u)\cdot\nabla_x\theta(t,x)\zeta_\delta(x) \\
&\quad + \eta'(u)S(t,x,u)\theta(t,x)\zeta_\delta(x)\,dt\,dx + \int_{(0,T)\times\partial\Omega} q(u_b)\cdot\hat{n}\,\theta(t,\hat{x})\,dt\,d\sigma.
\end{aligned}$$

Observe that

$$\int_Q q(u)\cdot\nabla_x\zeta_\delta(x)\theta(t,x)\,dt\,dx \xrightarrow{\delta\to 0} \int_0^T\int_{\partial\Omega} q\big(u^\tau\big)\cdot\hat{n}\,\theta(t,\hat{x})\,dt\,d\sigma,$$

$$\int_Q \eta'(u)S(t,x,u)\theta(t,x)\zeta_\delta(x)\,dt\,dx \xrightarrow{\delta\to 0} 0,$$

$$\int_Q \eta(u)\partial_t\theta(t,x)\zeta_\delta(x)\,dt\,dx \xrightarrow{\delta\to 0} 0,$$

and

$$\int_Q q\big(u^\varepsilon\big)\cdot\nabla_x\theta(t,x)\zeta_\delta(x)\,dt\,dx \xrightarrow{\delta\to 0} 0.$$

As a result,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} E_1 = \int_{(0,T)\times\partial\Omega} \big(q(u_b) - q\big(u^\tau\big)\big)\cdot\hat{n}\,\theta\,dt\,d\sigma.$$

Next,

$$E_2 := \varepsilon\int_Q \eta'\big(u^\varepsilon\big)\Delta_x u^\varepsilon\theta(t,x)\zeta_\delta(x)\,dt\,dx$$

$$
\begin{aligned}
&= \varepsilon \int_Q \left(\operatorname{div}_x\left(\eta'\left(u^\varepsilon\right)\nabla_x u^\varepsilon\right) - \eta''\left(u^\varepsilon\right)\left|\nabla_x u^\varepsilon\right|^2\right)\theta(t,x)\zeta_\delta(x)\,dt\,dx \\
&\leqslant \varepsilon \int_{(0,T)\times\partial\Omega} \eta'(u_b)\nabla_x u^\varepsilon \cdot \hat{n}\,\theta(t,\hat{x})\,dt\,d\sigma - \varepsilon \int_Q \eta'\left(u^\varepsilon\right)\nabla_x u^\varepsilon \cdot \nabla_x \zeta_\delta(x)\theta(t,x)\,dt\,dx \\
&\quad - \varepsilon \int_Q \eta'\left(u^\varepsilon\right)\nabla_x u^\varepsilon \cdot \nabla_x \theta(t,x)\zeta_\delta(x)\,dt\,dx \\
&=: E_{2,1} - E_{2,2} - E_{2,3}.
\end{aligned}
$$

Clearly, thanks to (37), $\lim_{\varepsilon\to 0}|E_{2,2}| = \lim_{\varepsilon\to 0}|E_{2,3}| = 0$.

To analyze $E_{2,1}$, we repeat the above argument with $\eta = \mathrm{Id}$ to obtain the equation

$$
\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left[\varepsilon \int_{(0,T)\times\partial\Omega} \nabla_x u^\varepsilon \cdot \hat{n}\,\Theta\,dt\,d\sigma\right] = \int_{(0,T)\times\partial\Omega} \left(A\left(u^\tau\right) - A(u_b)\right)\cdot \hat{n}\,\Theta\,dt\,d\sigma, \tag{42}
$$

which holds for all $\Theta \in C_c^\infty(\mathbb{R}^{d+1})$. Since the boundary $\partial\Omega$ is smooth, we can regularize (by mollification) u_b on $(0,T)\times\partial\Omega$; let us denote the regularized function by u_b^ς. By taking $\Theta = \eta'(u_b^\varsigma)\theta$ in (42) with $\theta \geqslant 0$ and sending $\varsigma \to 0$, we obtain

$$
\lim_{\varepsilon\to 0}\lim_{\delta\to 0} E_{2,1} = \int_{(0,T)\times\partial\Omega} \eta'(u_b)\left(A\left(u^\tau\right) - A(u_b)\right)\cdot \hat{n}\theta\,dt\,d\sigma.
$$

Hence, the limit u obeys the inequality

$$
\int_0^T \int_{\partial\Omega} \left[q\left(u^\tau\right) - q(u_b) - \eta'(u_b)\left(A\left(u^\tau\right) - A(u_b)\right)\right]\cdot \hat{n}\theta\,d\sigma\,dt \geqslant 0.
$$

By the arbitrariness of θ, the proof is complete. □

3.2. *Uniqueness proof*

In this section we prove the uniqueness part of Theorem 1.2, adapting the approach of Perthame [27,28]. In what follows, we let u, v denote two entropy solutions of the conservation law (1) with initial data $u_0, v_0 \in L^\infty$, respectively, and boundary data u_b, with the boundary condition (6) interpreted in the sense of (7). We start by rewriting the Dubois and LeFloch boundary condition (7) in a kinetic form due to Kwon [16].

Lemma 3.3. *The following two statements are equivalent*:

1. *For every convex entropy–entropy flux pair* (η, q),

$$
\left[q\left(u^\tau\right) - q(u_b) - \eta'(u_b)\left(A\left(u^\tau\right) - A(u_b)\right)\right]\cdot \hat{n} \geqslant 0 \quad \text{on } \Gamma.
$$

2. *There exists* $\mu \in \mathcal{M}^+(\Gamma \times (-L, L))$ *such that*

$$A'(\xi) \cdot \hat{n}\left[f^\tau(\hat{z}, \xi) - \chi\big(\xi; u_b(\hat{z})\big)\right] - \delta_{(\xi = u_b(\hat{z}))}\big(A\big(u^\tau\big) - A(u_b)\big) \cdot \hat{n} = -\partial_\xi \mu(\hat{z}, \xi),$$

for every $(\hat{z}, \xi) \in \Gamma \times (-L, L)$.

Associated with the entropy solutions u and v we introduce the corresponding χ-functions f and g defined by $f(t,x,\xi) = \chi(\xi; u(t,x))$ and $g(t,x,\xi) = \chi(\xi; v(t,x))$, respectively. In view of Theorem 2.1, there exist $m^1, m^2 \in \mathcal{M}^+(Q \times (-L, L))$ such that

$$\begin{aligned} &\partial_t f + A'(\xi) \cdot \nabla_x f + S(t,x,\xi)\big(\partial_\xi f - \delta(\xi)\big) = \partial_\xi m^1, \\ &\partial_t g + A'(\xi) \cdot \nabla_x g + S(t,x,\xi)\big(\partial_\xi g - \delta(\xi)\big) = \partial_\xi m^2. \end{aligned} \tag{43}$$

The goal is to show the following inequality for a.e. $t \in (0, T)$:

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \int_{-L}^{L} \big|f(t,x,\xi) - g(t,x,\xi)\big|^2 \, d\xi \, dx + \int_{\partial\Omega} \int_{-L}^{L} A'(\xi) \cdot \hat{n} \, \big|f^\tau(t,\hat{x},\xi) - g^\tau(t,\hat{x},\xi)\big|^2 \, d\xi \, d\sigma(\hat{x}) \\ &\quad \leqslant C \int_\Omega \big|S\big(t,x,u(t,x)\big) - S\big(t,x,v(t,x)\big)\big| \, dx, \end{aligned} \tag{44}$$

where $d\sigma$ denotes the volume element of $\partial\Omega$ and some constants $C > 0$.

To this end, we need to regularize f and g with respect to the t, x variables. Set $\epsilon = (\epsilon_1, \epsilon_2)$ and define ϕ_ϵ by

$$\phi_\epsilon(t,x) = \frac{1}{\epsilon_1} \phi_1\left(\frac{t}{\epsilon_1}\right) \frac{1}{\epsilon_2^d} \phi_2\left(\frac{x}{\epsilon_2}\right),$$

where $\phi_1 \in C_c^\infty(\mathbb{R})$, $\phi_2 \in C_c^\infty(\mathbb{R}^d)$ verify $\phi_j \geqslant 0$, $\int \phi_j = 1$ for $j = 1, 2$, and $\operatorname{supp}(\phi_1) \subset (-1, 0)$. We shall employ the following notations:

$$\begin{aligned} f_\epsilon(t,x,\xi) &= f(\cdot,\cdot,\xi) \overset{(t,x)}{\star} \phi_\epsilon(t,x), & g_\epsilon(t,x,\xi) &= g(\cdot,\cdot,\xi) \overset{(t,x)}{\star} \phi_\epsilon(t,x), \\ m_\epsilon^1(t,x,\xi) &= m^1(\cdot,\cdot,\xi) \overset{(t,x)}{\star} \phi_\epsilon(t,x), & m_\epsilon^2(t,x,\xi) &= m^2(\cdot,\cdot,\xi) \overset{(t,x)}{\star} \phi_\epsilon(t,x), \end{aligned}$$

where $\star$ means convolution with respect to the indicated variables and the mappings f, g, m_1, m_2 are extended to $\mathbb{R}^{d+1}$ by letting them take the value zero on $\mathbb{R}^{d+1} \setminus Q$.

The proof of the following lemma can be found in Perthame [27,28].

Lemma 3.4. *Let* m^1 *and* m^2 *be non-negative measures given in the Theorem* 2.1. *Then, the following holds*

$$\lim_{\epsilon \to 0} \int_{-L}^{L} m_\epsilon^1(\cdot,\cdot,\xi)\delta_{(\xi=u)} * \phi_\epsilon + m_\epsilon^2(\cdot,\cdot,\xi)\delta_{(\xi=v)} * \phi_\epsilon d\xi = 0 \quad \text{in } \mathcal{D}'(Q).$$

Let us continue with the proof of (44). Fix a $\partial\Omega$-regular deformation $\hat{\psi}$, and let Ω_s denote the open subset of Ω whose boundary is $\partial\Omega_s = \hat{\psi}(\{s\} \times \partial\Omega)$. Taking the convolution of each of the two kinetic equations in (43) and then subtracting the resulting equations we obtain an equation that is multiplied by $f_\epsilon - g_\epsilon$. The final outcome reads

$$\begin{aligned}
&\int_{\Omega_s} \int_{-L}^{L} \partial_t \big| f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi) \big|^2 + A'(\xi) \cdot \nabla_x \big| f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi) \big|^2 \, d\xi \, d\sigma_s \\
&\qquad + \int_{\Omega_s} \int_{-L}^{L} \big[S(t,x,\xi) \big(\partial_\xi (f-g) \big) \big] \overset{(t,x)}{\star} \phi_\epsilon(t,x) \big(f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi) \big) \, d\xi \, d\sigma_s \\
&\quad = 2 \int_{\Omega_s} \int_{-L}^{L} \partial_\xi \big(m_\epsilon^1(t,x,\xi) - m_\epsilon^2(t,x,\xi) \big) \big(f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi) \big) \, d\xi \, d\sigma_s, \qquad (45)
\end{aligned}$$

for a.e. $s > 0$, where $d\sigma_s$ denotes the volume element of $\partial\Omega_s$.

In view of Lemma 3.4, observe that for a.e. $s > 0$ we have

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int_{\Omega_s} \int_{-L}^{L} \partial_\xi \big(m_\epsilon^1(\cdot,\cdot,\xi) - m_\epsilon^2(\cdot,\cdot,\xi) \big) \big(f_\epsilon(\cdot,\cdot,\xi) - g_\epsilon(\cdot,\cdot,\xi) \big) \, d\xi \, d\sigma_s \\
&\quad = - \lim_{\epsilon \to 0} \int_{\Omega_s} \int_{-L}^{L} \big(m_\epsilon^1(\cdot,\cdot,\xi) - m_\epsilon^2(\cdot,\cdot,\xi) \big) \partial_\xi \big(f_\epsilon(\cdot,\cdot,\xi) - g_\epsilon(\cdot,\cdot,\xi) \big) \, d\xi \, d\sigma_s \\
&\quad = - \lim_{\epsilon \to 0} \int_{\Omega_s} \int_{-L}^{L} m_\epsilon^1(\cdot,\cdot,\xi) \delta_{(\xi = v)} \overset{(t,x)}{\star} \phi_\epsilon + m_\epsilon^2(\cdot,\cdot,\xi) \delta_{(\xi = u)} \overset{(t,x)}{\star} \phi_\epsilon \, d\xi \, d\sigma_s \leqslant 0.
\end{aligned}$$

Next, observe that

$$\begin{aligned}
&\limsup_{\epsilon \to 0} \left| \int_{\Omega_s} \int_{-L}^{L} \big[S(t,x,\xi) \big(\partial_\xi (f-g) \big) \big] \overset{(t,x)}{\star} \phi_\epsilon(t,x) \big(f_\epsilon(t,x,\xi) - g_\epsilon(t,x,\xi) \big) \, d\xi \, dx \right| \\
&\quad \leqslant 2 \int_{\Omega_s} \big| S(t,x,u) - S(t,x,v) \big| \, dx \\
&\quad \leqslant 2C \int_{\Omega_s} |u - v| \, dx, \quad \text{for a.e. } s > 0,
\end{aligned}$$

where we have used condition (3) to derive the last inequality. Indeed, using $|f| \leqslant 1$ and $|g| \leqslant 1$, we obtain $|f_\epsilon - g_\epsilon| \leqslant 2$ and we check that for $a.e.$ $(t,x) \in (0,T) \times \Omega$,

$$\int_{-L}^{L} \left[S(t,x,\xi)\big(\partial_\xi(f-g)\big)\right] \overset{(t,x)}{\star} \phi_\epsilon(t,x)\,d\xi \xrightarrow{\epsilon\to 0} S(t,x,v)-S(t,x,u),$$

thanks to $\partial_\xi(f-g)=\delta(\xi-v)-\delta(\xi-u)$.

Let us now apply the divergence theorem in (45) and subsequently take the limits $\epsilon\to 0$ and $s\to 0$. Applying Theorem 1.1 and the observations above, we obtain the following inequality for a.e. $t\in(0,T)$:

$$\int_{\Omega}\int_{-L}^{L} \partial_t\big|f(t,x,\xi)-g(t,x,\xi)\big|^2\,d\xi\,dx + \int_{\partial\Omega}\int_{-L}^{L} A'(\xi)\cdot\hat n\,\big|f^\tau(t,\hat x,\xi)-g^\tau(t,\hat x,\xi)\big|^2\,d\xi\,d\sigma(\hat x)$$
$$\leqslant 2\int_{\Omega}\big|S(t,x,u)-S(t,x,v)\big|\,dx. \tag{46}$$

Next, we show that the "boundary" part of (46) is non-negative. According to Lemma 3.3, there exist two measures $\mu_f,\mu_g\in\mathcal{M}^+(\Gamma\times(-L,L))$ corresponding to f and g, respectively, verifying

$$\begin{aligned}
&A'(\xi)\cdot\hat n\big[f^\tau(\hat z,\xi)-\chi\big(u_b(\hat z);\xi\big)\big]-\delta_{(\xi=u_b(\hat z))}\big(A\big(u^\tau(\hat z)\big)-A\big(u_b(\hat z)\big)\big)\cdot\hat n\\
&\quad=-\partial_\xi\mu_f(\hat z,\xi),\quad \text{for } (\hat z,\xi)\in\Gamma\times(-L,L),\\
&A'(\xi)\cdot\hat n\big[g^\tau(\hat z,\xi)-\chi\big(u_b(\hat z);\xi\big)\big]-\delta_{(\xi=u_b(\hat z))}\big(A\big(v^\tau(\hat z)\big)-A\big(u_b(\hat z)\big)\big)\cdot\hat n\\
&\quad=-\partial_\xi\mu_g(\hat z,\xi),\quad \text{for } (\hat z,\xi)\in\Gamma\times(-L,L),
\end{aligned}\tag{47}$$

where $\hat z$ means $(t,\hat x)$ and $\hat x\in\partial\Omega$. For later use, we notice that the mappings $\xi\mapsto\mu_f(\hat z,\xi)$, $\xi\mapsto\mu_g(\hat z,\xi)$ are continuous in $L^1(\Gamma)$ away from $\xi=u_b$.

For later use, observe that

$$\begin{aligned}
&A'\cdot\hat n\big|f^\tau-g^\tau\big|^2\\
&\quad=A'\cdot\hat n\big(f^\tau-\chi(u_b;\xi)\big)\operatorname{sgn}(\xi-u_b)-2A'\cdot\hat n\big(f^\tau-\chi(u_b;\xi)\big)\big(g^\tau-\chi(u_b;\xi)\big)\\
&\qquad+A'\cdot\hat n\,\big(g^\tau-\chi(u_b;\xi)\big)\operatorname{sgn}(\xi-u_b)\\
&\quad=A'\cdot\hat n\,\big(f^\tau-\chi(u_b;\xi)\big)\big[\operatorname{sgn}(\xi-u_b)-g^\tau+\chi(u_b;\xi)\big]\\
&\qquad+A'\cdot\hat n\,\big(g^\tau-\chi(u_b;\xi)\big)\big[\operatorname{sgn}(\xi-u_b)-f^\tau+\chi(u_b;\xi)\big]\\
&\quad=:A'\cdot\hat n\,\big(f^\tau-\chi(u_b;\xi)\big)\alpha(\hat z,\xi)+A'\cdot\hat n\,\big(g^\tau-\chi(u_b;\xi)\big)\beta(\hat z,\xi),
\end{aligned}\tag{48}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function, $\operatorname{sgn}(0)=0$.

Combining (47) and (48) gives

$$\int_{\partial\Omega}\int_{-L}^{L} A'(\xi)\cdot\hat n\big|f^\tau(\hat z,\xi)-g^\tau(\hat z,\xi)\big|^2\,d\xi\,d\sigma$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\int_{\partial\Omega}\Biggl(\int_{-L}^{u_b-\epsilon} + \int_{u_b+\epsilon}^{L}\Biggr) A'(\xi)\cdot\hat n\bigl[f^\tau(\hat z,\xi)-\chi\bigl(u_b(\hat z);\xi\bigr)\bigr]\alpha(\hat z,\xi)\,d\xi\,d\sigma \\
&\quad + \lim_{\epsilon\to 0}\int_{\partial\Omega}\Biggl(\int_{-L}^{u_b-\epsilon} + \int_{u_b+\epsilon}^{L}\Biggr) A'(\xi)\cdot\hat n\bigl[g^\tau(\hat z,\xi)-\chi\bigl(u_b(\hat z);\xi\bigr)\bigr]\beta(\hat z,\xi)\,d\xi\,d\sigma \\
&= \lim_{\epsilon\to 0}\int_{\partial\Omega}\Biggl(\int_{-L}^{u_b-\epsilon} + \int_{u_b+\epsilon}^{L}\Biggr)\bigl[-\partial_\xi\mu_f(\hat z,\xi)\alpha(\hat z,\xi)-\partial_\xi\mu_g(\hat z,\xi)\beta(\hat z,\xi)\bigr]\,d\xi\,d\sigma \\
&\geqslant \liminf_{\epsilon\to 0} I_\epsilon + \liminf_{\epsilon\to 0} J_\epsilon,
\end{aligned}
\tag{49}
$$

where

$$
I_\epsilon = \int_{\partial\Omega}\Biggl(\int_{-L}^{u_b-\epsilon} + \int_{u_b+\epsilon}^{L}\Biggr)\bigl[-\partial_\xi\mu_f(\hat z,\xi)\alpha(\hat z,\xi)\bigr]\,d\xi\,d\sigma \tag{50}
$$

and

$$
J_\epsilon = \int_{\partial\Omega}\Biggl(\int_{-L}^{u_b-\epsilon} + \int_{u_b+\epsilon}^{L}\Biggr)\bigl[-\partial_\xi\mu_g(\hat z,\xi)\beta(\hat z,\xi)\bigr]\,d\xi\,d\sigma. \tag{51}
$$

We claim that (49) is non-negative. Let us first prove that $\liminf I_\epsilon \geqslant 0$. To this end, we need to write the exact form of α. In fact, since $g(s,\cdot,\cdot)\to g^\tau$ in L^1_{loc} as $s\to 0$ and $g^\tau=\chi(v^\tau;\xi)$, we get $g^\tau(\hat z,\xi)=\chi(v^\tau(\hat z);\xi)$ for a.e. $(\hat z,\xi)\in\Gamma\times[-L,L]$ and thus it follows that $\alpha(\hat z,\xi)=\operatorname{sgn}(\xi-u_b)-\chi(v^\tau;\xi)+\chi(u_b;\xi)$. More explicitly, we have the following cases to consider:

Case 1. $u_b>0$.

(1) If $0<v^\tau\leqslant u_b$, then

$$
\alpha(\hat z,\xi)=\begin{cases} 1, & \text{if } \xi>u_b,\\ 0, & \text{if } v^\tau\leqslant\xi\leqslant u_b,\\ -1 & \text{if } \xi<v^\tau.\end{cases}
$$

(2) If $0<u_b<v^\tau$, then

$$
\alpha(\hat z,\xi)=\begin{cases} 1, & \text{if } \xi>v^\tau,\\ 0, & \text{if } u_b\leqslant\xi\leqslant v^\tau,\\ -1, & \text{if } \xi<u_b.\end{cases}
$$

(3) If $v^\tau\leqslant 0<u_b$, then

$$
\alpha(\hat z,\xi)=\begin{cases} 1, & \text{if } \xi>u_b,\\ 0, & \text{if } v^\tau\leqslant\xi\leqslant u_b,\\ -1, & \text{if } \xi<v^\tau.\end{cases}
$$

Case 2. $u_b \leqslant 0$.

(1) If $u_b \leqslant v^\tau \leqslant 0$, then

$$\alpha(\hat{z}, \xi) = \begin{cases} 1, & \text{if } \xi > v^\tau, \\ 0, & \text{if } u_b \leqslant \xi \leqslant v^\tau, \\ -1, & \text{if } \xi < u_b. \end{cases}$$

(2) If $v^\tau < u_b \leqslant 0$, then

$$\alpha(\hat{z}, \xi) = \begin{cases} 1, & \text{if } \xi > u_b, \\ 0, & \text{if } v^\tau \leqslant \xi \leqslant u_b, \\ -1 & \text{if } \xi < v^\tau. \end{cases}$$

(3) If $u_b \leqslant 0 < v^\tau$, then

$$\alpha(\hat{z}, \xi) = \begin{cases} 1, & \text{if } \xi > v^\tau, \\ 0, & \text{if } u_b \leqslant \xi \leqslant v^\tau, \\ -1, & \text{if } \xi < u_b. \end{cases}$$

Stated more compactly,

$$\alpha(\hat{z}, \xi) = \begin{cases} 1, & \text{if } \xi > \max\{u_b, v^\tau\}, \\ 0, & \text{if } \xi \in [\min\{u_b, v^\tau\}, \max\{u_b, v^\tau\}], \\ -1, & \text{if } \xi < \min\{u_b, v^\tau\}, \end{cases} \tag{52}$$

for a.e. $(\hat{z}, \xi) \in \Gamma \times [-L, L]$. Inserting (52) into (50) yields

$$I_\epsilon = \begin{cases} \int_{\partial\Omega} (\mu_f(\hat{z}, v^\tau) + \mu_f(\hat{z}, u_b - \epsilon))\, d\sigma, & \text{if } v^\tau \geqslant u_b, \\ \int_{\partial\Omega} (\mu_f(\hat{z}, v^\tau) + \mu_f(\hat{z}, u_b + \epsilon))\, d\sigma, & \text{if } v^\tau < u_b, \end{cases}$$

where we have taken into account that $\mu_f(\hat{z}, -L) = \mu_f(\hat{z}, L) = 0$. Therefore, we conclude that $\liminf I_\epsilon \geqslant 0$. Similarly, we prove that $\liminf J_\epsilon \geqslant 0$, cf. (51).

Let us now conclude the proof of Theorem 1.2. Since the second term in (46) is non-negative, Gronwall's inequality implies that for each fixed $\tau \in (0, t)$

$$\int_\Omega \int_{-L}^{L} \left| f(t, x, \xi) - g(t, x, \xi) \right|^2 d\xi\, dx \leqslant \exp(2CT) \int_\Omega \int_{-L}^{L} \left| f(\tau, x, \xi) - g(\tau, x, \xi) \right|^2 d\xi\, dx,$$

where C is given in (3).

Therefore, in view of Theorem 1.1, we can let $\tau \to 0$ to obtain

$$\int_\Omega \left| u(t, x) - v(t, x) \right| dx \leqslant \exp(2CT) \int_\Omega \left| u_0(x) - v_0(x) \right| dx, \quad \text{for a.e. } t \in (0, T).$$

This concludes the proof of Theorem 1.2.

4. IBVP for the Degasperis–Procesi equation

The purpose of this section is to prove Theorem 1.3. The main step of the proof relates to the existence of an entropy solution. Our existence argument is based passing to the limit in a vanishing viscosity approximation of (13).

Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [4]:

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon, & (t,x) \in (0,T) \times (0,1), \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \dfrac{3}{2} u_\varepsilon^2, & (t,x) \in (0,T) \times (0,1), \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in (0,1), \\ u_\varepsilon(t,0) = g_{\varepsilon,0}(t), \quad u_\varepsilon(t,1) = g_{\varepsilon,1}(t), & t \in (0,T), \\ \partial_x P_\varepsilon(t,0) = \psi_{\varepsilon,0}(t), \quad \partial_x P_\varepsilon(t,1) = \psi_{\varepsilon,1}(t), & t \in (0,T), \end{cases} \tag{53}$$

where $u_{\varepsilon,0}, g_{\varepsilon,0}, g_{\varepsilon,1}$ are C^∞ approximations of u_0, g_0, g_1, respectively, such that

$$g_{\varepsilon,0}(0) = u_{\varepsilon,0}(0), \qquad g_{\varepsilon,1}(0) = u_{\varepsilon,0}(1),$$

and

$$\psi_{\varepsilon,0} = -g'_{\varepsilon,0} - g_{\varepsilon,0} h_{\varepsilon,0}, \qquad \psi_{\varepsilon,1} = -g'_{\varepsilon,1} - g_{\varepsilon,1} h_{\varepsilon,1}. \tag{54}$$

Due to (54) and the first equation in (53), we have that

$$\partial_{xx}^2 u_\varepsilon(t,0) = \partial_{xx}^2 u_\varepsilon(t,1) = 0, \quad t \in (0,T). \tag{55}$$

For our own convenience let us convert (53) into a problem with homogeneous boundary conditions. To this end, we introduce the following notations:

$$\begin{aligned} \omega_\varepsilon(t,x) &= x g_{\varepsilon,1}(t) + (1-x) g_{\varepsilon,0}(t), \qquad v_\varepsilon = u_\varepsilon - \omega_\varepsilon, \\ \Omega_\varepsilon(t,x) &= \frac{x^2}{2} \psi_{\varepsilon,1}(t) + \frac{2x - x^2}{2} \psi_{\varepsilon,0}(t), \qquad V_\varepsilon = P_\varepsilon - \Omega_\varepsilon. \end{aligned} \tag{56}$$

Thanks to

$$\begin{aligned} \omega_\varepsilon(t,0) &= g_{\varepsilon,0}(t), \qquad \omega_\varepsilon(t,1) = g_{\varepsilon,1}(t), \quad t \in (0,T), \\ \partial_x \Omega_\varepsilon(t,0) &= \psi_{\varepsilon,0}(t), \qquad \partial_x \Omega_\varepsilon(t,1) = \psi_{\varepsilon,1}(t), \quad t \in (0,T), \end{aligned}$$

we have that

$$v_\varepsilon(t,0) = v_\varepsilon(t,1) = \partial_x V_\varepsilon(t,0) = \partial_x V_\varepsilon(t,1) = 0, \quad t \in (0,T). \tag{57}$$

Moreover, due to the definition of ω_ε and (55)

$$\partial_{xx}^2 \omega_\varepsilon(t,x) = \partial_{xxx}^3 \Omega_\varepsilon(t,x) = \partial_{xx}^2 v_\varepsilon(t,1) = \partial_{xx}^2 v_\varepsilon(t,0) = 0, \tag{58}$$

for each $t \in (0,T)$ and $x \in (0,1)$.

Finally, in view of (53) and (58), we obtain

$$\partial_t v_\varepsilon + \partial_t \omega_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 v_\varepsilon, \tag{59}$$

$$-\partial_{xx}^2 V_\varepsilon + V_\varepsilon = \frac{3}{2} u_\varepsilon^2 + \partial_{xx}^2 \Omega_\varepsilon - \Omega_\varepsilon. \tag{60}$$

We are now ready to state and prove our key estimate.

Lemma 4.1. *For each $t \in (0, T)$,*

$$\begin{aligned} &\left\| v_\varepsilon(t,\cdot)\right\|_{L^2(0,1)}^2 + 2\varepsilon e^{2\alpha_\varepsilon(t)} \int_0^t e^{-2\alpha_\varepsilon(s)} \left\| \partial_x v_\varepsilon(s,\cdot)\right\|_{L^2(0,1)}^2 ds \\ &\quad \leqslant 4\left\| v_\varepsilon(0,\cdot)\right\|_{L^2(0,1)}^2 e^{2\alpha_\varepsilon(t)} + 8 e^{2\alpha_\varepsilon(t)} \int_0^t e^{-2\alpha_\varepsilon(s)} \beta_\varepsilon(s)\, ds, \end{aligned} \tag{61}$$

where

$$\alpha_\varepsilon(t) = C_0 \left(t + \int_0^t \left(\left| g_{\varepsilon,0}(s)\right| + \left| g_{\varepsilon,1}(s)\right| \right) ds \right), \tag{62}$$

$$\begin{aligned} \beta_\varepsilon(t) = C_0 \Big(&\left| g_{0,\varepsilon}'(t)\right|^2 + \left| g_{1,\varepsilon}'(t)\right|^2 + \left| h_{0,\varepsilon}(t) g_{0,\varepsilon}(t)\right|^2 + \left| h_{1,\varepsilon}(t) g_{1,\varepsilon}(t)\right|^2 \\ &+ \left| g_{0,\varepsilon}(t)\right|^3 + \left| g_{1,\varepsilon}(t)\right|^3 \Big), \end{aligned} \tag{63}$$

and $C_0 > 0$ is a positive constant independent on ε.

In particular, the families

$$\{u_\varepsilon\}_{\varepsilon>0}, \qquad \{\sqrt{\varepsilon}\, \partial_x u_\varepsilon\}_{\varepsilon>0}$$

are bounded in $L^\infty(0, T; L^2(0, 1))$ and $L^2((0, T) \times (0, 1))$, respectively.

Proof. Following [5] we introduce the quantity $\theta_\varepsilon = \theta_\varepsilon(t, x)$ solving the following elliptic problem:

$$\begin{cases} -\partial_{xx}^2 \theta_\varepsilon + 4\theta_\varepsilon = v_\varepsilon(t,x), & x \in (0,1), \\ \theta_\varepsilon(t,0) = \theta_\varepsilon(t,1) = 0, & t \in (0,T). \end{cases} \tag{64}$$

Our motivation for bringing in (64) comes from the fact that, in the case of homogeneous boundary conditions, the quantity

$$\int_0^1 v_\varepsilon \left(\theta_\varepsilon - \partial_{xx}^2 \theta_\varepsilon \right) dx$$

is conserved by (8) when $\varepsilon = 0$ (see [10]). Thanks to (64) we have

$$\begin{aligned}\left\|\theta_\varepsilon(t,\cdot)\right\|_{H^2(0,1)} &\leqslant \left\|v_\varepsilon(t,\cdot)\right\|_{L^2(0,1)} \leqslant 4\left\|\theta_\varepsilon(t,\cdot)\right\|_{H^2(0,1)},\\ \left\|\partial_x\theta_\varepsilon(t,\cdot)\right\|_{H^2(0,1)} &\leqslant \left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^2(0,1)} \leqslant 4\left\|\partial_x\theta_\varepsilon(t,\cdot)\right\|_{H^2(0,1)}.\end{aligned} \tag{65}$$

Indeed, squaring both sides of (64),

$$v_\varepsilon^2 = \left(\partial_{xx}^2\theta_\varepsilon\right)^2 - 8\theta_\varepsilon\partial_x\theta_\varepsilon + 16\theta_\varepsilon^2$$

and integrating over (0, 1),

$$\begin{aligned}\int_0^1 v_\varepsilon^2\,dx &= \int_0^1 \left[\left(\partial_{xx}^2\theta_\varepsilon\right)^2 + 8(\partial_x\theta_\varepsilon)^2 + 16\theta_\varepsilon^2\right]dx + 8[\theta_\varepsilon\partial_x\theta_\varepsilon]_0^1\\ &= \int_0^1 \left[\left(\partial_{xx}^2\theta_\varepsilon\right)^2 + 8(\partial_x\theta_\varepsilon)^2 + 16\theta_\varepsilon^2\right]dx.\end{aligned}$$

Since

$$\begin{aligned}\int_0^1 \left[\left(\partial_{xx}^2\theta_\varepsilon\right)^2 + (\partial_x\theta_\varepsilon)^2 + \theta_\varepsilon^2\right]dx &\leqslant \int_0^1 \left[\left(\partial_{xx}^2\theta_\varepsilon\right)^2 + 8(\partial_x\theta_\varepsilon)^2 + 16\theta_\varepsilon^2\right]dx\\ &\leqslant 16\int_0^1 \left[\left(\partial_{xx}^2\theta_\varepsilon\right)^2 + (\partial_x\theta_\varepsilon)^2 + \theta_\varepsilon^2\right]dx,\end{aligned}$$

we have the first line of (65). For the second line in (65), since

$$\partial_{xx}^2\theta_\varepsilon(t,0) = \partial_{xx}^2\theta_\varepsilon(t,1) = 0 \quad \text{(cf. (57))},$$

we can argue in the same way.

We multiply (59) by $\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon$ and then integrate the result over (0, 1), obtaining

$$\begin{aligned}&\underbrace{\int_0^1 \partial_t v_\varepsilon\left(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)dx}_{A_1} + \underbrace{\int_0^1 \partial_t \omega_\varepsilon\left(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)dx}_{A_2} + \underbrace{\int_0^1 u_\varepsilon\partial_x u_\varepsilon\left(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)dx}_{A_3}\\ &\quad + \underbrace{\int_0^1 \partial_x P_\varepsilon\left(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)dx}_{A_4} = \varepsilon\underbrace{\int_0^1 \partial_{xx}^2 v_\varepsilon\left(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)dx}_{A_5}.\end{aligned} \tag{66}$$

Thanks to (57) and (64),

$$\begin{aligned}
A_1 &= \int_0^1 \partial_t\left(4\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)\left(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\right)dx \\
&= \int_0^1 \left(4\partial_t\theta_\varepsilon\theta_\varepsilon - 4\partial_t\theta_\varepsilon\partial_{xx}^2\theta_\varepsilon - \partial_{txx}^3\theta_\varepsilon\theta_\varepsilon + \partial_{txx}^3\theta_\varepsilon\partial_{xx}^2\theta_\varepsilon\right)dx \\
&= \int_0^1 \left(4\partial_t\theta_\varepsilon\theta_\varepsilon + 5\partial_{tx}^2\theta_\varepsilon\partial_x\theta_\varepsilon + \partial_{txx}^3\theta_\varepsilon\partial_{xx}^2\theta_\varepsilon\right)dx - \left[4\partial_t\theta_\varepsilon\partial_x\theta_\varepsilon + \partial_{tx}^2\theta_\varepsilon\theta_\varepsilon\right]_0^1 \\
&= \frac{1}{2}\frac{d}{dt}\int_0^1 \left(4\theta_\varepsilon^2 + 5(\partial_x\theta_\varepsilon)^2 + \left(\partial_{xx}^2\theta_\varepsilon\right)^2\right)dx = \frac{1}{2}\frac{d}{dt}\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2,
\end{aligned} \tag{67}$$

where

$$\|f\|_{\widetilde{H}^2(0,1)} = \sqrt{4\|f\|_{L^2(0,1)}^2 + 5\|f'\|_{L^2(0,1)}^2 + \|f''\|_{L^2(0,1)}^2}.$$

The Hölder inequality, (11), and (56) guarantee that

$$\begin{aligned}
A_2 &\leqslant \int_0^1 (\partial_t\omega_\varepsilon)^2\,dx + \frac{1}{2}\int_0^1 \theta_\varepsilon^2\,dx + \frac{1}{2}\int \left(\partial_{xx}^2\theta_\varepsilon\right)^2 dx \\
&\leqslant 2\left(\left|g_{0,\varepsilon}'(t)\right|^2 + \left|g_{1,\varepsilon}'(t)\right|^2\right) + \frac{1}{2}\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2.
\end{aligned} \tag{68}$$

In light of (56), (57), and (60),

$$\begin{aligned}
A_4 &= \int_0^1 \left(\partial_x V_\varepsilon\theta_\varepsilon - \partial_x V_\varepsilon\partial_{xx}^2\theta_\varepsilon + \partial_x\Omega_\varepsilon\theta_\varepsilon - \partial_x\Omega_\varepsilon\partial_{xx}^2\theta_\varepsilon\right)dx \\
&= \int_0^1 \left(\partial_x V_\varepsilon\theta_\varepsilon + \partial_{xx}^2 V_\varepsilon\partial_x\theta_\varepsilon + \partial_x\Omega_\varepsilon\theta_\varepsilon - \partial_x\Omega_\varepsilon\partial_{xx}^2\theta_\varepsilon\right)dx - \left[\partial_x V_\varepsilon\partial_x\theta_\varepsilon\right]_0^1 \\
&= \int_0^1 \left(\partial_x\left(V_\varepsilon - \partial_{xx}^2 V_\varepsilon\right)\theta_\varepsilon + \partial_x\Omega_\varepsilon\theta_\varepsilon - \partial_x\Omega_\varepsilon\partial_{xx}^2\theta_\varepsilon\right)dx + \left[\partial_{xx}^2 V_\varepsilon\theta_\varepsilon\right]_0^1 \\
&= \int_0^1 \left(3u_\varepsilon\partial_x u_\varepsilon\theta_\varepsilon - \partial_x\Omega_\varepsilon\partial_{xx}^2\theta_\varepsilon\right)dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
A_3+A_4 &= \int_0^1 \big(u_\varepsilon \partial_x u_\varepsilon \big(4\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\big) - \partial_x \Omega_\varepsilon \partial_{xx}^2\theta_\varepsilon\big)\,dx \\
&= \int_0^1 \big(u_\varepsilon \partial_x u_\varepsilon v_\varepsilon - \partial_x \Omega_\varepsilon \partial_{xx}^2\theta_\varepsilon\big)\,dx \\
&= \int_0^1 \big(u_\varepsilon^2 \partial_x u_\varepsilon - u_\varepsilon \partial_x u_\varepsilon \omega_\varepsilon - \partial_x \Omega_\varepsilon \partial_{xx}^2\theta_\varepsilon\big)\,dx \\
&= \int_0^1 \left(\frac{u_\varepsilon^2}{2}\partial_x \omega_\varepsilon - \partial_x \Omega_\varepsilon \partial_{xx}^2\theta_\varepsilon\right)dx + \left[\frac{u_\varepsilon^3}{3} - \frac{u_\varepsilon^2}{2}\omega_\varepsilon\right]_0^1 \\
&\leqslant \frac{|g_{0,\varepsilon}(t)|+|g_{1,\varepsilon}(t)|}{2}\int_0^1 u_\varepsilon^2\,dx + \frac{1}{2}\int_0^1 \big(\partial_{xx}^2\theta_\varepsilon\big)^2\,dx + \frac{1}{2}\int_0^1 (\partial_x\Omega_\varepsilon)^2\,dx \\
&\quad + \frac{|g_{0,\varepsilon}(t)|^3+|g_{1,\varepsilon}(t)|^3}{6} \\
&\leqslant c_1\big(\big|g_{0,\varepsilon}(t)\big| + \big|g_{1,\varepsilon}(t)\big| + 1\big)\big\|\theta_\varepsilon(t,\cdot)\big\|_{\widetilde{H}^2(0,1)}^2 \\
&\quad + c_1\big(\big|\psi_{0,\varepsilon}(t)\big|^2 + \big|\psi_{1,\varepsilon}(t)\big|^2 + \big|g_{0,\varepsilon}(t)\big|^3 + \big|g_{1,\varepsilon}(t)\big|^3\big) \\
&\leqslant c_1\big(\big|g_{0,\varepsilon}(t)\big| + \big|g_{1,\varepsilon}(t)\big| + 1\big)\big\|\theta_\varepsilon(t,\cdot)\big\|_{\widetilde{H}^2(0,1)}^2 \\
&\quad + c_1\big(\big|g_{0,\varepsilon}'(t)\big|^2 + \big|g_{1,\varepsilon}'(t)\big|^2 + \big|h_{0,\varepsilon}(t)g_{0,\varepsilon}(t)\big|^2 + \big|h_{1,\varepsilon}(t)g_{1,\varepsilon}(t)\big|^2 \\
&\quad + \big|g_{0,\varepsilon}(t)\big|^3 + \big|g_{1,\varepsilon}(t)\big|^3\big),
\end{aligned} \tag{69}$$

for some constant $c_1>0$ that is independent on ε.

By observing that (57) and (64) furnish

$$\partial_{xx}^2\theta_\varepsilon(t,0) = \partial_{xx}^2\theta_\varepsilon(t,1) = 0, \quad t\in(0,T),$$

we achieve

$$\begin{aligned}
A_5 &= \varepsilon\int_0^1 \partial_{xx}^2\big(4\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\big)\big(\theta_\varepsilon - \partial_{xx}^2\theta_\varepsilon\big)\,dx \\
&= \varepsilon\int_0^1 \big(4\partial_{xx}^2\theta_\varepsilon\theta_\varepsilon - 4\big(\partial_{xx}^2\theta_\varepsilon\big)^2 - \partial_{xxxx}^4\theta_\varepsilon\theta_\varepsilon + \partial_{xxxx}^4\theta_\varepsilon\partial_{xx}^2\theta_\varepsilon\big)\,dx
\end{aligned}$$

$$= \varepsilon \int_0^1 \left(-4(\partial_x \theta_\varepsilon)^2 - 4\left(\partial_{xx}^2 \theta_\varepsilon\right)^2 + \partial_{xxx}^3 \theta_\varepsilon \partial_x \theta_\varepsilon - \left(\partial_{xxx}^3 \theta_\varepsilon\right)^2\right) dx$$

$$+ \varepsilon\left[4\partial_x \theta_\varepsilon \theta_\varepsilon - \partial_{xxx}^3 \theta_\varepsilon \theta_\varepsilon + \partial_{xxx}^3 \theta_\varepsilon \partial_{xx}^2 \theta_\varepsilon\right]_0^1$$

$$= -\varepsilon \int_0^1 \left(4(\partial_x \theta_\varepsilon)^2 + 5\left(\partial_{xx}^2 \theta_\varepsilon\right)^2 + \left(\partial_{xxx}^3 \theta_\varepsilon\right)^2\right) dx + \varepsilon\left[\partial_{xx}^2 \theta_\varepsilon \partial_x \theta_\varepsilon\right]_0^1$$

$$= -\varepsilon \left\|\partial_x \theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2. \tag{70}$$

In view of (67), (68), (69), and (70), it follows from (66) that

$$\begin{aligned}
&\frac{d}{dt}\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 + 2\varepsilon\left\|\partial_x \theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 \\
&\quad \leqslant c_2\left(\left|g_{0,\varepsilon}(t)\right| + \left|g_{1,\varepsilon}(t)\right| + 1\right)\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 \\
&\qquad + c_2\left(\left|g_{0,\varepsilon}'(t)\right|^2 + \left|g_{1,\varepsilon}'(t)\right|^2 + \left|h_{0,\varepsilon}(t) g_{0,\varepsilon}(t)\right|^2 + \left|h_{1,\varepsilon}(t) g_{1,\varepsilon}(t)\right|^2 \right. \\
&\qquad \left. + \left|g_{0,\varepsilon}(t)\right|^3 + \left|g_{1,\varepsilon}(t)\right|^3\right),
\end{aligned} \tag{71}$$

for some constant $c_2 > 0$ that is independent on ε.

Using the notations introduced in (62) and (63), inequality (71) becomes

$$\frac{d}{dt}\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 + 2\varepsilon\left\|\partial_x \theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 \leqslant \alpha_\varepsilon'(t)\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 + \beta_\varepsilon(t),$$

and hence, thanks to the Gronwall lemma,

$$\begin{aligned}
&\left\|\theta_\varepsilon(t,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 + 2\varepsilon e^{\alpha_\varepsilon(t)} \int_0^t e^{-\alpha_\varepsilon(s)}\left\|\partial_x \theta_\varepsilon(s,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 ds \\
&\quad \leqslant \left\|\theta_\varepsilon(0,\cdot)\right\|_{\widetilde{H}^2(0,1)}^2 e^{\alpha_\varepsilon(t)} + 2e^{\alpha_\varepsilon(t)} \int_0^t e^{-\alpha_\varepsilon(s)} \beta_\varepsilon(s)\, ds.
\end{aligned} \tag{72}$$

Clearly, via (65), the desired claim (61) follows from (72).

The boundedness of the families $\{u_\varepsilon\}_{\varepsilon>0}$, $\{\sqrt{\varepsilon}\partial_x u_\varepsilon\}_{\varepsilon>0}$ follows from the definition of the auxiliary variable v_ε in (56) and assumption (11). □

We continue with some a priori bounds that come directly from the energy estimate stated in Lemma 4.1.

Lemma 4.2. *The families $\{V_\varepsilon\}_{\varepsilon>0}$, $\{P_\varepsilon\}_{\varepsilon>0}$ are both bounded in*

$$L^\infty\left(0,T; W^{2,1}(0,1)\right) \cap L^\infty\left(0,T; W^{1,\infty}(0,1)\right).$$

In particular, these families are bounded in $L^\infty((0,T)\times(0,1))$.

Proof. To simplify the notation, let us introduce the quantity

$$f_\varepsilon = \frac{3}{2}u_\varepsilon^2 + \partial_{xx}^2 \Omega_\varepsilon - \Omega_\varepsilon.$$

From (57) and (60),

$$-\partial_{xx}^2 V_\varepsilon + V_\varepsilon = f_\varepsilon, \qquad \partial_x V_\varepsilon(t,0) = \partial_x V_\varepsilon(t,1) = 0.$$

Using the function

$$G(x,y) = \begin{cases} \frac{e^x+e^{-x}}{2}\frac{e^{y-1}+e^{1-y}}{e-e^{-1}}, & \text{if } 0 \leqslant x \leqslant y \leqslant 1,\\ \frac{e^y+e^{-y}}{2}\frac{e^{x-1}+e^{1-x}}{e-e^{-1}}, & \text{if } 0 \leqslant y \leqslant x \leqslant 1, \end{cases}$$

which is the Green's function of the operator $1 - \partial_{xx}^2$ on $(0,1)$ with homogeneous Neumann boundary conditions at $x = 0, 1$, we have the formulas

$$V_\varepsilon(t,x) = \int_0^1 G(x,y) f_\varepsilon(t,y)\,dy, \qquad \partial_x V_\varepsilon(t,x) = \int_0^1 \partial_x G(x,y) f_\varepsilon(t,y)\,dy. \tag{73}$$

Since $G \geqslant 0$ and $G, \partial_x G \in L^\infty((0,1)\times(0,1))$, we can estimate as follows:

$$\left|V_\varepsilon(t,x)\right| \leqslant \int_0^1 G(x,y)\left|f_\varepsilon(t,y)\right| dy \leqslant \|G\|_{L^\infty((0,1)^2)} \left\|f(t,\cdot)\right\|_{L^1(0,1)},$$

$$\left|\partial_x V_\varepsilon(t,x)\right| \leqslant \int_0^1 \left|\partial_x G(x,y)\right|\left|f_\varepsilon(t,y)\right| dy \leqslant \|\partial_x G\|_{L^\infty((0,1)^2)} \left\|f(t,\cdot)\right\|_{L^1(0,1)},$$

$$\left\|\partial_{xx}^2 V_\varepsilon(t,\cdot)\right\|_{L^1(0,1)} \leqslant \left\|V_\varepsilon(t,\cdot)\right\|_{L^1(0,1)} + \left\|f_\varepsilon(t,\cdot)\right\|_{L^1(0,1)}.$$

Thanks to Lemma 4.1, we conclude that the desired bounds on $\{V_\varepsilon\}_{\varepsilon>0}$ hold.

Finally, the bounds on $\{P_\varepsilon\}_{\varepsilon>0}$ follow from the bounds on $\{V_\varepsilon\}_{\varepsilon>0}$ and (11). □

Using the previous lemma we can bound u_ε and v_ε in L^∞ (cf. [7, Lemma 4]).

Lemma 4.3. *For every $t \in (0,T)$,*

$$\left\|u_\varepsilon(t,\cdot)\right\|_{L^\infty(0,1)} \leqslant \|u_0\|_{L^\infty(0,1)} + \|g_0\|_{L^\infty(0,T)} + \|g_1\|_{L^\infty(0,T)} + C_T t,$$

for some constant $C_T > 0$ depending on T but not on ε.

Proof. Due to (53) and Lemma 4.2,

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon \leqslant \sup_{\varepsilon>0} \|\partial_x P_\varepsilon\|_{L^\infty((0,T)\times(0,1))} \leqslant C_T.$$

Since the map

$$f(t) := \|u_0\|_{L^\infty(0,1)} + \|g_0\|_{L^\infty(0,T)} + \|g_1\|_{L^\infty(0,T)} + C_T t, \quad t \in (0,T),$$

solves the equation

$$\frac{df}{dt} = C_T$$

and

$$u_\varepsilon(0,x), g_0(t), g_1(t) \leqslant f(t), \qquad (t,x) \in (0,T)\times(0,1),$$

the comparison principle for parabolic equations implies that

$$u_\varepsilon(t,x) \leqslant f(t), \quad (t,x) \in (0,T)\times(0,1).$$

In a similar way we can prove that

$$u_\varepsilon(t,x) \geqslant -f(t), \quad (t,x) \in (0,T)\times(0,1).$$

This concludes the proof of the lemma. □

As a consequence of Lemmas 4.2 and 4.3, the second equation in (53) yields.

Lemma 4.4. *The families* $\{V_\varepsilon\}_{\varepsilon>0}$, $\{P_\varepsilon\}_{\varepsilon>0}$ *are bounded in* $L^\infty(0,T;W^{2,\infty}(0,1))$.

Let us continue by proving the existence of a distributional solution to (8), (9), (10) satisfying (18).

Lemma 4.5. *There exists a function* $u \in L^\infty((0,T)\times(0,1))$ *that is a distributional solution of* (13) *and satisfies* (18) *in the sense of distributions for every convex entropy* $\eta \in C^2(\mathbb{R})$.

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations (53). We use the compensated compactness method [30].

Lemma 4.6. *There exist a subsequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a limit function* $u \in L^\infty((0,T)\times(0,1))$ *such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p\big((0,T)\times(0,1)\big),\ 1 \leqslant p < \infty. \tag{74}$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and let $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q'(u) = \eta'(u)\, u$. By multiplying the first equation in (53) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{=:\mathcal{L}_\varepsilon^1} \underbrace{-\varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 + \eta'(u_\varepsilon)\partial_x P_\varepsilon}_{=:\mathcal{L}_\varepsilon^2},$$

where $\mathcal{L}_\varepsilon^1$, $\mathcal{L}_\varepsilon^2$ are distributions. By Lemmas 4.1–4.4,

$$\begin{gathered} \mathcal{L}_\varepsilon^1 \to 0 \quad \text{in } H^{-1}\big((0,T) \times (0,1)\big), \\ \mathcal{L}_\varepsilon^2 \text{ is uniformly bounded in } L^1\big((0,T) \times (0,1)\big). \end{gathered} \tag{75}$$

Therefore, Murat's lemma [22] implies that

$$\big\{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\big\}_{\varepsilon>0} \text{ lies in a compact subset of } H_{\mathrm{loc}}^{-1}\big((0,T) \times (0,1)\big). \tag{76}$$

The L^∞ bound stated in Lemma 4.3, (76), and the Tartar's compensated compactness method [30] give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $u \in L^\infty((0,T) \times (0,1))$ such that (74) holds. □

Lemma 4.7. *We have*

$$P_{\varepsilon_k} \to P^u \quad \text{in } L^p\big(0,T; W^{1,p}(0,1)\big),\ 1 \leqslant p < \infty, \tag{77}$$

where the sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and the function u are constructed in Lemma 4.6*.*

Proof. Using the integral representation of V_{ε_k} stated in (73), Lemma 4.3, and arguing as in [5, Theorem 3.2], we get

$$\begin{aligned} &\big\| P_{\varepsilon_k} - P^u \big\|_{L^p(0,T;W^{1,p}(0,1))} \\ &\quad \leqslant C\big(\|u_{\varepsilon_k} - u\|_{L^p((0,T)\times(0,1))} + \|\psi_{\varepsilon_k,1} - \psi_1\|_{L^p(0,T)} + \|\psi_{\varepsilon_k,0} - \psi_0\|_{L^p(0,T)}\big), \end{aligned}$$

for every $1 \leqslant p < \infty$ and some constant $C > 0$ depending on u_0, g_0, g_1, but not on ε. Therefore Lemma 4.6 gives (77). □

Proof of Lemma 4.5. Fix a test function $\phi \in C_c^\infty([0,T) \times [0,1])$. Due to (53)

$$\begin{aligned} &\int_0^T \int_0^1 \left(u_\varepsilon \partial_t \phi + \frac{u_\varepsilon^2}{2} \partial_x \phi - \partial_x P_\varepsilon \phi + \varepsilon u_\varepsilon \partial_{xx}^2 \phi\right) dx\, dt + \int_0^1 u_{0,\varepsilon}(x) \phi(0,x)\, dx \\ &\quad + \int_0^T g_{0,\varepsilon}(t)\phi(t,0)\, dt - \int_0^T g_{1,\varepsilon}(t)\phi(t,1)\, dt = 0. \end{aligned}$$

Therefore, by the assumptions on $u_{0,\varepsilon}$, $g_{0,\varepsilon}$, $g_{1,\varepsilon}$ and Lemmas 4.6, 4.7, we conclude that the function u constructed in Lemma 4.6 is a distributional solution of (13).

Finally, we have to verify that the distributional solution u satisfies the entropy inequality stated in (18). Let $\eta \in C^2(\mathbb{R})$ be a convex entropy. The convexity of η and (53) yield

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon)\partial_x P_\varepsilon \leqslant \varepsilon \partial_{xx}^2 \eta(u_\varepsilon).$$

Therefore, (18) follows from Lemmas 4.6 and 4.7. □

We are now ready for the proof of Theorem 1.3.

Proof of Theorem 1.3. Since, thanks to Lemma 4.5, $u \in L^\infty((0,T)\times(0,1))$ is a distributional solution of the problem

$$\begin{cases} \partial_t u + u\partial_x u = -\partial_x P^u, & (t,x) \in (0,T)\times(0,1), \\ u(0,x) = u_0(x), & x \in (0,1), \\ u(t,0) = g_0(t), \quad u(t,1) = g_1(t), & t \in (0,T), \end{cases} \tag{78}$$

that satisfies the entropy inequalities (18), Theorem 1.1 tells us that the limit u admits strong boundary traces u_0^τ, u_1^τ at $(0,T)\times\{x=0\}$, $(0,T)\times\{x=1\}$, respectively. Since, arguing as in Section 3.1 (indeed our solution is obtained as the vanishing viscosity limit of (78)), Lemma 3.2 and the boundedness of the source term $\partial_x P^u$ (cf. (17)) imply (19).

Finally, we have to prove the uniqueness of the entropy solution to (8), (9), (10). To this end, let u_1, u_2 be two entropy solutions. We have to prove that

$$u_1 = u_2 \text{ a.e. in } (0,T)\times(0,1). \tag{79}$$

Since u_1 and u_2 are entropy solutions of (78), we can slightly modify the arguments in Subsection 3.2 to account for two different (nonlocal) source terms $S^1 := \partial_x P^{u_1}$ and $S^2 := \partial_x P^{u_2}$, or apply the result of [1, Corollary 2.6], to assemble the inequality

$$\left\| u_1(t,\cdot) - u_2(t,\cdot)\right\|_{L^1(0,1)} \leqslant c \left\| \partial_x P^{u_1} - \partial_x P^{u_2}\right\|_{L^1((0,t)\times(0,1))}, \tag{80}$$

for $t \in (0,T)$ and a constant c. Moreover, (16) says that

$$\partial_x P^{u_1}(t,x) - \partial_x P^{u_2}(t,x) = \frac{3}{2}\int_0^1 \partial_x G(x,y)\big(u_1(t,y)+u_2(t,y)\big)\big(u_1(t,y)-u_2(t,y)\big)\,dy.$$

Hence, by Lemma 4.3 and (80),

$$\left\| u_1(t,\cdot) - u_2(t,\cdot)\right\|_{L^1(0,1)} \leqslant C\|u_1 - u_2\|_{L^1((0,t)\times(0,1))}, \quad t \in (0,T),$$

for some constant $C > 0$. Therefore, (79) follows from Gronwall's lemma. □

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